A REMARK ON COMMUTABLE FUNCTIONS AND CONTINUOUS ITERATIONS

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The purpose of this note is to apply some recent results of B. Choczewski to a question concerning commutable functions and continuous iterations. Namely, we shall prove the following:

**Theorem.** Let \( f(x) \) be a function of class \( C^2 \) in an interval \( (a, b) \), such that \( f(x) > x \) in \( (a, b) \), \( f(b) = b \), \( \lim_{x \to a+0} f(x) = a \), \( f'(x) > 0 \) in \( (a, b) \), and \( f'(b) < 1 \). Moreover let us assume that the function \( (d^2/dx^2)f^{-1}(x) \) satisfies a Lipschitz condition in a left neighbourhood of the point \( b \) (\( f^{-1}(x) \) denotes the inverse function of \( f(x) \), which exists since \( f'(x) > 0 \)).

Then there exists a unique one-parameter family of functions \( \phi_s(x) \), \( s \in (-\infty, \infty) \), satisfying the following conditions for all \( s \):

(A) \( \phi_s(x) \) is of class \( C^2 \) in \( (a, b) \).

(B) \( \phi_s(x) \in (a, b) \) for \( x \in (a, b) \), \( \phi_s(b) = b \).

(C) \( \phi_s(x) \) commutes with \( f(x) \), i.e.

\[
\phi_s(f(x)) = f(\phi_s(x)).
\]

The family of the functions \( \phi_s(x) \) is also, in a sense, the "best" family of continuous iterations of the function \( f(x) \). The functions \( \phi_s(x) \) may be obtained as the limit of the sequence defined by

\[
\phi_{s,n}(x) = f^{-n}(\phi_{s,0}[f^n(x)]), \quad x \in (a, b), \quad n = 1, 2, 3, \ldots,
\]

where \( f^k(x) \) denotes the \( k \)th iterate of the function \( f(x) \), i.e., \( f^0(x) = x \), \( f^{k+1}(x) = f[f^k(x)] \), \( f^{k-1}(x) = f^{-1}[f^k(x)] \), \( k = 0, \pm 1, \pm 2, \ldots \), and \( \phi_{s,0}(x) \) is an arbitrary function of class \( C^2 \) in \( (a, b) \) such that

\[
\phi_{s,0}(x) \in (a, b) \text{ for } x \in (a, b), \quad \phi_{s,0}^\prime(b) = f^\prime(b)^s.
\]

Furthermore, if the function \( f(x) \) is of class \( C^r(r > 2) \) in \( (a, b) \), then the functions \( \phi_s(x) \) are also of class \( C^r \) in \( (a, b) \).

**Proof.** In order to simplify the notation we write

\[
h(x) = f^{-1}(x).
\]
Then equation (1) takes the form

\[ \phi(x) = h(\phi[f(x)]). \]  

Moreover, we have

\[ h'(b)[f'(b)]^2 = f'(b) < 1. \]  

Consequently, as a direct consequence of a theorem of B. Choczewski [2], it follows that for every number \( s \in (-\infty, \infty) \) there exists exactly one function \( \phi_s(x) \) satisfying equation (4), and fulfilling conditions (A), (B), and the condition

\[ \phi_s'(b) = [f'(b)]^s. \]  

(Condition (C) is fulfilled, since the functions \( \phi_s(x) \) satisfy equation (4), which is equivalent to (1)).

It follows from the uniqueness of \( \phi_s \) that \( \phi_{k}(x) = f^k(x) \) for integral \( k \).

In particular we have \( \phi_1(x) = f(x) \).

The paper by B. Choczewski has not yet been published; and since the method employed by him is essential to our further considerations, we shall present here a very short outline of his arguments, naturally omitting all details.

Let \( \mathfrak{A} \) be the space of all functions \( \phi(x) \) of class \( C^2 \) in the interval \( (b - \eta, b) \) (where \( \eta \) is a sufficiently small, fixed positive number, depending only on \( f(x) \) and \( s \)), fulfilling the conditions:

\[ \phi(x) \in \langle a + \alpha, b \rangle \quad \text{for} \quad x \in \langle b - \eta, b \rangle, \quad \phi(b) = b, \quad \phi'(b) = [f'(b)]^s. \]

\[ (\text{where } \alpha \text{ and } K \text{ are suitably chosen positive numbers, depending only on } f(x) \text{ and } s \), \text{ and endowed with the metric } \]

\[ \rho[\phi, \psi] = \sup_{(b - \eta, b)} |\phi''(x) - \psi''(x)|, \]

(with which \( \mathfrak{A} \) becomes a complete metric space). Consider the transformation

\[ \Phi[\phi] = h(\phi[f(x)]). \]

This transformation (under a suitable choice of \( \eta, \alpha \) and \( K \)) maps \( \mathfrak{A} \)

\[ * \]

\[ Of \text{ course, for } s \in (-\infty, \infty) \text{ the values of } [f'(b)]^s \text{ run over the set of all positive numbers. However, in order that } \phi_s(x) \text{ form a family of continuous iterations of } f(x) \text{ (cf. (9) below), it is convenient to define } \phi_s(x) \text{ with the aid of relation (6)}. \]

\[ * \]

\[ In \text{ his paper [2]} \text{ B. Choczewski deals with the more general equation } \phi(x) = H(x, \phi[f(x)]). \]
into itself. Furthermore, in view of (5), we have, for $\phi, \psi \in \mathfrak{R}$

$$\rho[\Phi[\phi], \Phi[\psi]] < \theta \rho[\phi, \psi], \quad \theta < 1.$$ 

Consequently, $\Phi$ is a contraction mapping, and by the theorem of Banach-Caccioppoli, has exactly one fixed point in $\mathfrak{R}$, i.e. there exists exactly one function $\Phi_t(x)$ of class $C^2$ in $(b-\eta, b)$, satisfying equation (4) and conditions (7). This function can be uniquely extended onto the whole interval $(a, b)$ in such a manner that it will satisfy equation (4).

Hence it follows (see also [3, Theorem V]) that the function $\Phi_t(x)$ can be obtained as the limit of the sequence defined by the recurrence formula

$$(8) \quad \phi_{s,n+1}(x) = h(\phi_{s,n}[f(x)]), \quad n = 0, 1, 2, \ldots$$

where $\phi_{s,0}(x)$ is an arbitrary function which, when restricted to $(b-\eta, b)$, belongs to $\mathfrak{R}$, and thus e.g. an arbitrary function of class $C^2$ in $(a, b)$, fulfilling conditions (3). One can easily verify that formula (2) follows immediately from (8).

From the theorems of B. Choczewski [2] it further follows that the function $\Phi_t(x)$ has the same regularity properties as $f(x)$.

To complete the proof of our theorem it remains only to show that the functions $\Phi_t(x)$ form a family of continuous iterations of $f(x)$, i.e. that they satisfy the relation

$$(9) \quad \phi_t[\phi_s(x)] = \phi_{s+t}(x), \quad x \in (a, b), \quad s, t \in (-\infty, \infty).$$

Now, the function $\phi_t[\phi_s(x)]$, just like $\phi_s(x)$ and $\phi_t(x)$, is of class $C^2$ in $(a, b)$ and evidently commutes with $f(x)$. We have also $\phi_t[\phi_s(x)] \\
\in (a, b)$ for $x \in (a, b)$, $\phi_s[\phi_t(b)] = b$, $(d/dx)\phi_t[\phi_t(x)]\big|_{x=b} = \phi_t[\phi_t(b)]\phi_t'(b) = \phi_t'(b)\phi_t'(b) = [f'(b)]^{s+t}$. But the function with these properties is unique. Thus $\phi_t[\phi_t(x)] = \phi_{t+s}(x)$, which was to be proved.

**Remark.** The above theorem is closely related to some results of L. Berg [1]. The results obtained by Berg are, in some respect, stronger than ours—the condition $b < \infty$ does not occur, and the case $f'(b) = 1$ is also discussed. On the other hand, in most of the theorems in [1] assumptions of analyticity, or at least of multiple differentiability of $f(x)$, as well as the assumption that $f''(x)$ has a constant sign, occur; while in our case the assumptions regarding the regularity of $f(x)$ are weaker.

Similar equations have been also investigated by S. Sternberg [5]. He has obtained similar results, however in a different way.

It follows from [4] that in condition (A) it is not sufficient to re-
quire only the continuity of $\phi'(x)$: for equation (1) admits continuous solution depending on an arbitrary function. Unfortunately, however, our method does not allow us to decide whether in addition to $\phi'(x)$ there also exist other functions of class $C^1$ in $(a, b)$ and fulfilling conditions (B) and (C). (For the linear function; $f(x) = b + q(x = b)$ such functions do not exist; cf. [1, §1].)

References

3. J. Kordylewski and M. Kuczma, On the functional equation $F(x, \phi(x), \phi[f(x)]) = 0$, Ann. Polon. Math. 7 (1959), 21–32.

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