A REMARK ON COMMUTABLE FUNCTIONS 
AND CONTINUOUS ITERATIONS 

MAREK KUCZMA

The purpose of this note is to apply some recent results of B. Choczewski to a question concerning commutable functions and continuous iterations. Namely, we shall prove the following:

THEOREM. Let \( f(x) \) be a function of class \( C^2 \) in an interval \((a, b)\), such that \( f(x) > x \) in \((a, b)\), \( f(b) = b \), \( \lim_{x \to a+} f(x) = a \), \( f'(x) > 0 \) in \((a, b)\), and \( f'(b) < 1 \). Moreover let us assume that the function \( (d^2/dx^2)f^{-1}(x) \) satisfies a Lipschitz condition in a left neighbourhood of the point \( b \) (\( f^{-1}(x) \) denotes the inverse function of \( f(x) \), which exists since \( f'(x) > 0 \)).

Then there exists a unique one-parameter family of functions \( \phi_s(x) \), \( s \in (-\infty, \infty) \), satisfying the following conditions for all \( s \):

(A) \( \phi_s(x) \) is of class \( C^2 \) in \((a, b)\).
(B) \( \phi_s(x) \in (a, b) \) for \( x \in (a, b) \), \( \phi_s(b) = b \).
(C) \( \phi_s(x) \) commutes with \( f(x) \), i.e.

\[
\phi_s(f(x)) = f(\phi_s(x)).
\]

The family of the functions \( \phi_s(x) \) is also, in a sense, the "best" family of continuous iterations of the function \( f(x) \). The functions \( \phi_s(x) \) may be obtained as the limit of the sequence defined by

\[
\phi_{s,n}(x) = f^{-n}(\phi_{s,0}[f^n(x)]), \quad x \in (a, b), \quad n = 1, 2, 3, \ldots,
\]

where \( f^n(x) \) denotes the \( k \)th iterate of the function \( f(x) \), i.e., \( f^0(x) = x \), \( f^{k+1}(x) = f[f^k(x)] \), \( f^{-1}(x) = f^{-1}[f(x)] \), \( k = 0, \pm 1, \pm 2, \ldots \), and \( \phi_{s,0}(x) \) is an arbitrary function of class \( C^2 \) in \((a, b)\) such that

\[
\phi_{s,0}(x) \in (a, b) \) for \( x \in (a, b) \), \( \phi_{s,0}(b) = b \), \( \phi_{s,0}'(b) = [f'(b)]^s \).
\]

Furthermore, if the function \( f(x) \) is of class \( C^r(r > 2) \) in \((a, b)\), then the functions \( \phi_s(x) \) are also of class \( C^r \) in \((a, b)\).

PROOF. In order to simplify the notation we write

\[
h(x) = f^{-1}(x).
\]

Received by the editors October 30, 1961.

1 The number \( a \) may also equal \( -\infty \); but \( b \) must be finite, which undoubtedly is a defect of the present theorem. The condition \( \lim_{x \to a+} f(x) = a \) may be omitted, but then the functions \( \phi_s(x) \) occurring in the thesis of the theorem may not be defined in the whole interval \((a, b)\), but only in a left neighbourhood of the point \( b \). The conditions \( f(x) > x \) in \((a, b)\), \( f'(b) < 1 \), may be replaced by \( f(x) < x \) in \((a, b)\), \( f'(b) > 1 \). And naturally, the endpoints, \( a \) and \( b \), of the interval in question may be interchanged.

847
Then equation (1) takes the form

\[(4) \quad \phi(x) = h(\phi[f(x)]).\]

Moreover, we have

\[(5) \quad h'(<b)[f'(b)]^2 = f'(b) < 1.\]

Consequently, as a direct consequence of a theorem of B. Choczewski [2], it follows that for every number \(s \in (-\infty, \infty)\) there exists exactly one function \(\phi_s(x)\) satisfying equation (4), and fulfilling conditions (A), (B), and the condition²

\[(6) \quad \phi_s'(b) = [f'(b)]^s.\]

(Condition (C) is fulfilled, since the functions \(\phi_s(x)\) satisfy equation (4), which is equivalent to (1)).

It follows from the uniqueness of \(\phi_s\) that \(\phi_k(x) = f^k(x)\) for integral \(k\). In particular we have \(\phi_1(x) = f(x)\).

The paper by B. Choczewski has not yet been published; and since the method employed by him is essential to our further considerations, we shall present here a very short outline of his arguments, naturally omitting all details.³

Let \(\mathfrak{a}\) be the space of all functions \(\phi(x)\) of class \(C^2\) in the interval \(\langle b - \eta, b \rangle\) (where \(\eta\) is a sufficiently small, fixed positive number, depending only on \(f(x)\) and \(s\)), fulfilling the conditions:

\[(7) \quad \phi(x) \in \langle a + \alpha, b \rangle \text{ for } x \in \langle b - \eta, b \rangle, \quad \phi(b) = b, \quad \phi'(b) = [f'(b)]^s, \quad |\phi''(x)| \leq K \quad \text{for } x \in \langle b - \eta, b \rangle,\]

(where \(\alpha\) and \(K\) are suitably chosen positive numbers, depending only on \(f(x)\) and \(s\)), and endowed with the metric

\[\rho[\phi, \psi] = \sup_{\langle b - \eta, b \rangle} |\phi''(x) - \psi''(x)|,\]

(with which \(\mathfrak{a}\) becomes a complete metric space). Consider the transformation

\[\Phi[\phi] = h(\phi[f(x)]).\]

This transformation (under a suitable choice of \(\eta, \alpha\) and \(K\)) maps \(\mathfrak{a}\)

² Of course, for \(s \in (-\infty, \infty)\) the values of \([f'(b)]^s\) run over the set of all positive numbers. However, in order that \(\phi_s(x)\) form a family of continuous iterations of \(f(x)\) (cf. (9) below), it is convenient to define \(\phi_s(x)\) with the aid of relation (6).

³ In his paper [2] B. Choczewski deals with the more general equation \(\phi(x) = H(x, \phi[f(x)]).\)
into itself. Furthermore, in view of (5), we have, for $\phi, \psi \in \mathcal{A}$

$$\rho[\Phi[\phi], \Phi[\psi]] < \vartheta \rho[\phi, \psi],$$

Consequently, $\Phi$ is a contraction mapping, and by the theorem of Banach-Caccioppoli, has exactly one fixed point in $\mathcal{A}$, i.e. there exists exactly one function $\phi_s(x)$ of class $C^2$ in $[b-\eta, b)$, satisfying equation (4) and conditions (7). This function can be uniquely extended onto the whole interval $(a, b)$ in such a manner that it will satisfy equation (4).

Hence it follows (see also [3, Theorem V]) that the function $\phi_s(x)$ can be obtained as the limit of the sequence defined by the recurrence formula

$$\phi_{s,n+1}(x) = h(\phi_{s,n}[f(x)]), \quad n = 0, 1, 2, \ldots,$$

where $\phi_{s,0}(x)$ is an arbitrary function which, when restricted to $[b-\eta, b)$, belongs to $\mathcal{A}$, and thus e.g. an arbitrary function of class $C^2$ in $(a, b)$, fulfilling conditions (3). One can easily verify that formula (2) follows immediately from (8).

From the theorems of B. Choczewski [2] it further follows that the function $\phi_s(x)$ has the same regularity properties as $f(x)$.

To complete the proof of our theorem it remains only to show that the functions $\phi_s(x)$ form a family of continuous iterations of $f(x)$, i.e. that they satisfy the relation

$$\phi_s[\phi_t(x)] = \phi_{s+t}(x), \quad x \in (a, b), \quad s, t \in (-\infty, \infty).$$

Now, the function $\phi_t[\phi_s(x)]$, just like $\phi_s(x)$ and $\phi_t(x)$, is of class $C^2$ in $(a, b)$ and evidently commutes with $f(x)$. We have also $\phi_s[\phi_t(x)] \in (a, b)$ for $x \in (a, b)$, $\phi_s[\phi_t(b)] = b$, $(d/dx)\phi_s[\phi_t(x)] \bigg|_{x=b} = \phi'_s[\phi_t(b)]\phi'_t(b) = \phi'_t(b)\phi'_t(b) = [f'(b)]^{s+t}$. But the function with these properties is unique. Thus $\phi_s[\phi_t(x)] = \phi_{s+t}(x)$, which was to be proved.

Remark. The above theorem is closely related to some results of L. Berg [1]. The results obtained by Berg are, in some respect, stronger than ours—the condition $b < \infty$ does not occur, and the case $f'(b) = 1$ is also discussed. On the other hand, in most of the theorems in [1] assumptions of analyticity, or at least of multiple differentiability of $f(x)$, as well as the assumption that $f''(x)$ has a constant sign, occur; while in our case the assumptions regarding the regularity of $f(x)$ are weaker.

Similar equations have been also investigated by S. Sternberg [5]. He has obtained similar results, however in a different way.

It follows from [4] that in condition (A) it is not sufficient to re-
quire only the continuity of \( \phi_s(x) \): for equation (1) admits continuous solution depending on an arbitrary function. Unfortunately, however, our method does not allow us to decide whether in addition to \( \phi_s(x) \) there also exist other functions of class \( C^1 \) in \((a, b)\) and fulfilling conditions (B) and (C). (For the linear function; \( f(x) = b + q(x = b) \) such functions do not exist; cf. [1, §1].)

References

3. J. Kordylewski and M. Kuczma, On the functional equation \( F(x, \phi(x), \phi[f(x)]) = 0 \), Ann. Polon. Math. 7 (1959), 21–32.

Uniwersytet Jagielloński, Kraków, Poland