A REMARK ON COMMUTABLE FUNCTIONS
AND CONTINUOUS ITERATIONS

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The purpose of this note is to apply some recent results of B. Choczewski to a question concerning commutable functions and continuous iterations. Namely, we shall prove the following:

**Theorem.** Let $f(x)$ be a function of class $C^2$ in an interval $(a, b)$, such that $f(x) > x$ in $(a, b)$, $f(b) = b$, $\lim_{x \to a^+} f(x) = a$, $f'(x) > 0$ in $(a, b)$, and $f'(b) < 1$. Moreover let us assume that the function $(d^2/dx^2)f^{-1}(x)$ satisfies a Lipschitz condition in a left neighbourhood of the point $b$ ($f^{-1}(x)$ denotes the inverse function of $f(x)$, which exists since $f'(x) > 0$).

Then there exists a unique one-parameter family of functions $\phi_s(x)$, $s \in (-\infty, \infty)$, satisfying the following conditions for all $s$:

(A) $\phi_s(x)$ is of class $C^2$ in $(a, b)$.

(B) $\phi_s(x) \in (a, b)$ for $x \in (a, b)$, $\phi_s(b) = b$.

(C) $\phi_s(x)$ commutes with $f(x)$, i.e.,

\[ f[\phi_s(x)] = \phi_s[f(x)]. \]

The family of the functions $\phi_s(x)$ is also, in a sense, the "best" family of continuous iterations of the function $f(x)$. The functions $\phi_s(x)$ may be obtained as the limit of the sequence defined by

\[ \phi_{s,n}(x) = f^{-n}(\phi_{s,0}[f^n(x)]), \quad x \in (a, b), \quad n = 1, 2, 3, \ldots, \]

where $f^n(x)$ denotes the $k$th iterate of the function $f(x)$, i.e., $f^0(x) = x$, $f^{k+1}(x) = f^n(f^k(x))$, $f^{k-1}(x) = f^{-1}[f^k(x)]$, $k = 0, \pm 1, \pm 2, \ldots$, and $\phi_{s,0}(x)$ is an arbitrary function of class $C^2$ in $(a, b)$ such that

\[ \phi_{s,0}(x) \in (a, b) \text{ for } x \in (a, b), \phi_{s,0}(b) = b, \phi_{s,0}'(b) = \left[f'(b)\right]^s. \]

Furthermore, if the function $f(x)$ is of class $C^r (r > 2)$ in $(a, b)$, then the functions $\phi_s(x)$ are also of class $C^r$ in $(a, b)$.

**Proof.** In order to simplify the notation we write

\[ h(x) = f^{-1}(x). \]

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1 The number $a$ may also equal $-\infty$; but $b$ must be finite, which undoubtedly is a defect of the present theorem. The condition $\lim_{x \to a^+} f(x) = a$ may be omitted, but then the functions $\phi_s(x)$ occurring in the thesis of the theorem may not be defined in the whole interval $(a, b)$, but only in a left neighbourhood of the point $b$. The conditions $f(x) > x$ in $(a, b)$, $f'(b) < 1$, may be replaced by $f(x) < x$ in $(a, b)$, $f'(b) > 1$. And naturally, the endpoints, $a$ and $b$, of the interval in question may be interchanged.
Then equation (1) takes the form
\[
\phi(x) = h(\phi[f(x)]).
\]
Moreover, we have
\[
(5) \quad h'(b) [f'(b)]^2 = f'(b) < 1.
\]
Consequently, as a direct consequence of a theorem of B. Choczewski [2], it follows that for every number \( s \in (−\infty, \infty) \) there exists exactly one function \( \phi_s(x) \) satisfying equation (4), and fulfilling conditions (A), (B), and the condition\(^2\)
\[
(6) \quad \phi'_s(b) = [f'(b)]^s.
\]
(Condition (C) is fulfilled, since the functions \( \phi_s(x) \) satisfy equation (4), which is equivalent to (1)).

It follows from the uniqueness of \( \phi_s \) that \( \phi_k(x) = f^k(x) \) for integral \( k \). In particular we have \( \phi_1(x) = f(x) \).

The paper by B. Choczewski has not yet been published; and since the method employed by him is essential to our further considerations, we shall present here a very short outline of his arguments, naturally omitting all details.\(^3\)

Let \( \mathfrak{A} \) be the space of all functions \( \phi(x) \) of class \( C^2 \) in the interval \( (b − \eta, b) \) (where \( \eta \) is a sufficiently small, fixed positive number, depending only on \( f(x) \) and \( s \)), fulfilling the conditions:
\[
(7) \quad \phi(x) \in (a + \alpha, b) \text{ for } x \in (b − \eta, b), \quad \phi(b) = b, \quad \phi'(b) = [f'(b)],
\]
\[
|\phi''(x)| \leq K, \quad \text{for } x \in (b − \eta, b),
\]
(where \( \alpha \) and \( K \) are suitably chosen positive numbers, depending only on \( f(x) \) and \( s \)), and endowed with the metric
\[
\rho[\phi, \psi] = \sup_{(b − \eta, b)} |\phi''(x) − \psi''(x)|,
\]
(with which \( \mathfrak{A} \) becomes a complete metric space). Consider the transformation
\[
\Phi[\phi] = h(\phi[f(x)]).
\]
This transformation (under a suitable choice of \( \eta, \alpha \) and \( K \)) maps \( \mathfrak{A} \)
\[^{2}\text{Of course, for } s \in (−\infty, \infty) \text{ the values of } [f'(b)]^s \text{ run over the set of all positive numbers. However, in order that } \phi_s(x) \text{ form a family of continuous iterations of } f(x) \text{ (cf. (9) below), it is convenient to define } \phi_s(x) \text{ with the aid of relation (6).}
\[^{3}\text{In his paper [2] B. Choczewski deals with the more general equation } \phi(x) = H(x, \phi[f(x)]).\]

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into itself. Furthermore, in view of (5), we have, for $\phi, \psi \in \mathcal{A}$

$$\rho[\Phi[\phi], \Phi[\psi]] < \theta \rho[\phi, \psi],$$

Consequently, $\Phi$ is a contraction mapping, and by the theorem of Banach-Caccioppoli, has exactly one fixed point in $\mathcal{A}$, i.e. there exists exactly one function $\phi_\ast(x)$ of class $C^2$ in $(b-\eta, b)$, satisfying equation (4) and conditions (7). This function can be uniquely extended onto the whole interval $(a, b)$ in such a manner that it will satisfy equation (4).

Hence it follows (see also [3, Theorem V]) that the function $\phi_\ast(x)$ can be obtained as the limit of the sequence defined by the recurrence formula

$$\phi_{n+1}(x) = h(\phi_n[f(x)]), \quad n = 0, 1, 2, \ldots,$$

where $\phi_{n,0}(x)$ is an arbitrary function which, when restricted to $(b-\eta, b)$, belongs to $\mathcal{A}$, and thus e.g. an arbitrary function of class $C^2$ in $(a, b)$, fulfilling conditions (3). One can easily verify that formula (2) follows immediately from (8).

From the theorems of B. Choczewski [2] it further follows that the function $\phi_\ast(x)$ has the same regularity properties as $f(x)$.

To complete the proof of our theorem it remains only to show that the functions $\phi_\ast(x)$ form a family of continuous iterations of $f(x)$, i.e. that they satisfy the relation

$$\phi_s[\phi_t(x)] = \phi_{s+t}(x), \quad x \in (a, b), \quad s, t \in (-\infty, \infty).$$

Now, the function $\phi_n[\phi_t(x)]$, just like $\phi_\ast(x)$ and $\phi_t(x)$, is of class $C^2$ in $(a, b)$ and evidently commutes with $f(x)$. We have also $\phi_n[\phi_t(x)] 
\in (a, b)$ for $x \in (a, b)$, $\phi_n[\phi_t(b)] = b$, $(d/dx)\phi_n[\phi_t(x)] \big|_{x=b} = \phi'_n[\phi_t(b)] \phi'_t(b) = \phi'_t(b) \phi_t'(b) = [f'(b)]^{n+t}$. But the function with these properties is unique. Thus $\phi_n[\phi_t(x)] = \phi_{s+t}(x)$, which was to be proved.

REMARK. The above theorem is closely related to some results of L. Berg [1]. The results obtained by Berg are, in some respect, stronger than ours—the condition $b < \infty$ does not occur, and the case $f'(b) = 1$ is also discussed. On the other hand, in most of the theorems in [1] assumptions of analyticity, or at least of multiple differentiability of $f(x)$, as well as the assumption that $f''(x)$ has a constant sign, occur; while in our case the assumptions regarding the regularity of $f(x)$ are weaker.

Similar equations have been also investigated by S. Sternberg [5]. He has obtained similar results, however in a different way.

It follows from [4] that in condition (A) it is not sufficient to re-
quire only the continuity of \( \phi(x) \): for equation (1) admits continuous solution depending on an arbitrary function. Unfortunately, however, our method does not allow us to decide whether in addition to \( \phi(x) \) there also exist other functions of class \( C' \) in \((a, b)\) and fulfilling conditions (B) and (C). (For the linear function; \( f(x) = b + q(x = b) \) such functions do not exist; cf. [1, §1].)

References

3. J. Kordylewski and M. Kuczma, On the functional equation \( F(x, \phi(x), \phi[f(x)] = 0 \), Ann. Polon. Math. 7 (1959), 21–32.