BOUNDEDNESS OF LIMITS

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This is an answer to a problem of A. Wilansky [2]. Suppose \( \sum_{i} b_{i} \) converges absolutely. Out of each infinite sequence \( x = \{ x_{n} \} \) we may construct

(1) \( \beta_{1}(x) = x_{1}, \quad \beta_{n}(x) = x_{n} + x_{n-1} + \sum_{i}^{n-1} b_{i} x_{i} \quad (n > 1). \)

Obviously \( ||x|| = \text{l.u.b.} \{ ||\beta_{1}(x)||, ||\beta_{2}(x)||, \cdots \} \) forms a norm over the linear space of all convergent sequences. The problem is whether the linear functional \( \lim x = \lim_{n \to \infty} x_{n} \) is bounded in this norm.

We may solve (1) to have

(2) \( x_{n} = \sum_{r=1}^{n} (-1)^{n-r} g_{r} \beta_{r}(x) \)

where

(3) \( g_{n} = 1, \quad g_{n} = \begin{bmatrix} 1+b_{r} & 1 & 0 & \cdots & 0 \\ b_{r} & 1+b_{r+1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{r} & b_{r+1} & b_{r+2} & \cdots & 1+b_{n-1} \end{bmatrix} \quad (r < n). \)

On expanding \( g_{n}^{(r)} \) along the diagonal we have \( g_{n}^{(r)} = 1 + \text{linear combination of products of } b_{r}, \cdots, b_{n-1} \) with factors of \( (1-b_{r})(1-b_{r+1}) \cdots (1-b_{n-1}) \) as coefficients. Hence

(4) \( |g_{n}^{(r)}| < B, \quad |g_{n}^{(r)} - 1| < C(|b_{r}| + \cdots + |b_{n-1}|) \)

where \( B, C \) are independent of \( n \) and \( r \).

By (3) we also have

(5) \( g_{n}^{(r+1)} - g_{n}^{(r+1)} = b_{r} g_{n}^{(r+1)} - b_{r} \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1+b_{r+2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & b_{r+2} & b_{r+3} & \cdots & 1+b_{n-1} \end{bmatrix}, \)

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1 Editorial note: After this paper had been accepted for publication a specific counterexample to Wilansky’s conjecture was published by Hayman and Wilansky[1].
\begin{align*}
g^{(r)}_n - g^{(r)}_{n-1} &= b^{(r)}_{n-1}g^{(r)}_{n-1} - b^{(r)}_{n-2}g^{(r)}_{n-2} + \cdots + (-1)^{n-r}b^{(r)}_{r+1}g^{(r)}_{r+1} \\
&\quad + (-1)^{n-r-1}b_r.
\end{align*}

Expanding (5) along the diagonal and arguing as in (4),
\begin{equation}
\left| g^{(r)}_n - g^{(r+1)}_n \right| < D \left| b_r \right|
\end{equation}
where \( D \) is independent of \( n \) and \( r \).

Let
\begin{align*}
z_1 &= x_1, \\
z_n &= x_n + x_{n-1}, \\
\beta_{n,r} &= \beta_r - \beta_{r+1} + \cdots + (-1)^{n-r}\beta_n,
\end{align*}
then
\begin{equation}
z_n = k^{(1)}_n \beta_1(x) + \cdots + k^{(n-1)}_n \beta_{n-1}(x) + \beta_n(x)
\end{equation}
where
\begin{align*}
k^{(r)}_n &= (-1)^{n-r}\frac{g^{(r)}_n}{g^{(r)}_{n-1}} + (-1)^{n-r-1}\frac{b_r}{g^{(r)}_{n-1}} = -b_r + b^{(r)}_{r+1}g^{(r)}_{r+1} - \cdots \\
&\quad + (-1)^{n-r}b^{(r)}_{n-1}g^{(r)}_{n-1}.
\end{align*}
Since \( \beta_{r+2+s+1,s} = x_{r-1} - x_{r+s+1} - (b_r x_r + \cdots + b_{r+s} x_{r+s}) \), we have
\begin{equation}
\beta_{n,r} < E
\end{equation}
where \( E \) is independent of \( n \) and \( r \). Then
\begin{align*}
\left| \sum_{s=1}^{n-1} k^{(r)}_s \beta_s(x) \right| &= \left| k^{(r+1)}_n \beta_{n,s+1} + \sum_{s=1}^{n-2} (k^{(r)}_n + k^{(r+1)}_n) \beta_{n,s} + k^{(n-1)}_n \beta_{n,s-1} \right| \\
&\leq E \left( 2C + 2D \left( \left| b_{s+1} \right| + \cdots + \left| b_{n-1} \right| \right) \\
&\quad + 2C \left| b_{n-1} \right| \right) .
\end{align*}
Hence, from (8),
\begin{equation}
\lim x = \frac{1}{2} \left[ \lim_{n \to \infty} \beta_n(x) + \sum_{r=0}^{\infty} k^{(r)}_r \beta_r(x) \right]
\end{equation}
where
\begin{align*}
k^{(r)}_n &= \lim_{n \to \infty} k^{(r)}_n = \sum_{s=0}^{\infty} (-1)^{s-1}g^{(r)}_{r+s}b^{(r)}_{r+s}.
\end{align*}

**Theorem.** The linear functional \( \lim \) is bounded if and only if \( \sum_{r=0}^{\infty} \left| k^{(r)} \right| < \infty \).

**Proof.** If \( \sum_{r=0}^{\infty} \left| k^{(r)} \right| < \infty \), then (11) implies
\begin{equation}
\left| \lim x \right| < \frac{1}{2} \left( 1 + \sum_{r=0}^{\infty} \left| k^{(r)} \right| \right) \left\| x \right\| .
\end{equation}
If $\sum_{i=1}^{\infty} |k^{(r)}| = \infty$, then for any positive number $M$ we have $|k^{(1)}| + |k^{(2)}| + \cdots + |k^{(m)}| = K > 4M$ for some $m$ large enough. Let $\beta_i = \left(2M \frac{|k^{(i)}|}{Kk^{(i)}}\right)$ for $k^{(i)} \neq 0$ and $i \leq m$, $\beta_i = 0$ for $k^{(i)} = 0$ or $i > m$. Construct $x_n$ and $z_n$ according to (2) and (8), then $\lim_{n \to \infty} x_n = 2M$ and by (9) whenever $n > m$, $|z_n - z_{n+1}| < (mB)|b_n|$. So $(z_2 - z_1) + (z_3 - z_2) + \cdots$ converges absolutely. Hence

$$x_{2n} = (z_2 - z_1) + (z_4 - z_3) + \cdots + (z_{2n} - z_{2n-1}) \to M + \xi,$$

$$x_{2n+1} = x_{2n+1} - x_{2n} \to M - \xi.$$

(i) $\xi = 0$. Then $\lim x = M$ and $\|x\| < 1$.

(ii) $\xi \neq 0$. There is an integer $p > m$ such that $|b_{p+1}| + |b_{p+2}| + \cdots < 1/2|\xi|$. Let

$$x'_n = x_n \quad (n \leq p),$$

$$x'_n = x_n + \xi \quad (n > p \text{ is odd}),$$

$$x'_n = x_n - \xi \quad (n > p \text{ is even}).$$

Then $\lim x' = M$ and $\|x'\| < 1$.

Hence $\lim$ is unbounded.

**Remark.** There are cases for both possibilities. Suppose $|b_k| = O(k^{-\alpha})$, $\alpha > 3/2$. It is not difficult to verify (using the second inequality of (4)) that $|k^{(r)}| = |b_r - b_{r+1} + \cdots| + O(r^{-2(\alpha-1)})$. Then $\sum_{r=1}^{\infty} |k^{(r)}|$ converges or diverges with $\sum_{r=1}^{\infty} |b_r - b_{r+1} + \cdots|$. 

**References**


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