THE RECIPROCAL OF A FOURIER SERIES

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Edrei and Szegö [1] have posed the following problem: *Given the Fourier coefficients of a function \( G(x) \) find the Fourier coefficients of the reciprocal of the function without actually evaluating \( G(x) \)*. They were able to solve this problem in the case that \( G(x) \geq 0 \). Unfortunately this restriction eliminates the interesting case of complex-valued functions such as those which arise in Laurent series.

In this note it is shown possible to obtain a solution without the restriction, \( G(x) \geq 0 \). This is achieved by first treating the more general problem of finding the coefficients of \( F(x)/G(x) \), given the coefficients of \( F(x) \) and \( G(x) \).

Edrei and Szegö confine attention to classical Fourier series. Their problem is treated here for arbitrary orthogonal series expansions.²

Let \( \theta_j(x) \), \( j = 1, 2, \cdots \), be a sequence of bounded orthonormal functions in some region \( R \) of a space \( S \). Thus

\[
\int_R \theta_j(x) \theta_k^*(x) \, dx = \delta_{jk}
\]

where the asterisk denotes the complex conjugate. The Fourier coefficients of a function \( F(x) \in L \) are denoted by \( f_j \) and are given by the integral

\[
f_j = \int_R F(x) \theta_j^*(x) \, dx, \quad j = 1, 2, \cdots.
\]

If \( f_j = 0 \) for all \( j \) it is assumed that \( F(x) = 0 \) almost everywhere. In other words the orthonormal sequence is closed in \( L \).

**Problem 1.** Given the Fourier coefficients \( f_n \) and \( g_n \) of two functions \( F(x) \) and \( G(x) \) find the Fourier coefficients \( h_n \) of the function \( H(x) = F(x)/G(x) \).

A related question concerns a direct determination of \( H(x) \) without first finding \( F(x) \) and \( G(x) \). This may be stated as

**Problem 2.** In terms of \( f_n \) and \( g_n \) find expansion coefficient sets \( (\rho_1^{(m)}, \rho_2^{(m)}, \cdots, \rho_m^{(m)}) \) for \( m = 1, 2, \cdots \) such that

\[
\lim_{m \to \infty} \int_R \left| H(x) - \sum_{k=1}^m \rho_k^{(m)} \theta_k(x) \right| \, dx = 0.
\]

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² This generalization was suggested to the writer by David Moskovitz.
Given a solution of Problem 2 then

$$\lim_{m \to \infty} \int_R \left[ H(x) - \sum_{k=1}^m \rho_k^{(m)} \theta_k(x) \right] \theta_j^*(x) dx = 0.$$  

This is seen to give

$$h_j = \lim_{m \to \infty} \rho_j^{(m)}$$

and consequently the solution of Problem 2 also furnishes a solution of Problem 1. (In the following $\rho_j^{(m)}$ is denoted by $\rho_j$.)

**PROBLEM 3.** The solution of Problem 2 is sought but given the “Fourier matrix”

$$g_{jk} = \int_R G(x) \theta_j^*(x) \theta_k(x) dx$$

instead of the Fourier coefficients $g_j$. In other words find $h_n$ in terms of $f_n$ and $g_{jk}$. The following formal considerations indicate the close relationship between Problem 2 and Problem 3. Let

$$\beta_{jk} = \int_R \theta_j^*(x) \theta_k(x) \theta_r(x) dx.$$  

Then

$$\theta_j^*(x) \theta_k(x) = \sum_{r=1}^\infty \beta_{jk} \theta_r(x).$$

This relation is multiplied by $G(x)$ and integrated termwise to yield

$$g_{jk} = \sum_{r=1}^\infty \beta_{jk} g_r.$$  

Thus if (9) is valid then a solution of Problem 3 will yield a solution of Problem 2.

**Theorem 1.** Let $G(x) \geq 0$, $G(x) \in L$, $1/G(x) \in L$, and $F^2(x)/G(x) \in L$. Then for each integer $m$ the equations

$$f_j = \sum_{k=1}^m g_{jk} \rho_k, \quad j = 1, \ldots, m,$$

may be solved for $\rho_1, \rho_2, \ldots, \rho_m$ and these are expansion coefficient sets for $H(x) = F(x)/G(x)$ and this solves Problem 3.
Proof. Since $F = HG^{1/2}G^{1/2}$ the Schwarz inequality gives

$$\left( \int |F| \, dx \right)^2 \leq \int |H^2G| \, dx \int |G| \, dx,$$

and

$$\left( \int |H| \, dx \right)^2 \leq \int |H^2G| \, dx \int |G^{-1}| \, dx.$$

This proves that $F \in L$ and $H \in L$. Then it is seen that the following integral exists for any choice of the constants $\rho_j$:

$$E = \int \left| H(x) - \sum_{k=1}^{m} \rho_k \theta_k(x) \right|^2 G(x) \, dx.$$

Since $G(x) \geq 0$ this may be written in the form

$$E = \int \left| HG^{1/2} - \sum_{k=1}^{m} \rho_k \theta_k G^{1/2} \right|^2 \, dx.$$

This leads to consideration of the relation

$$\int U(x) \theta_j^*(x) G^{1/2} \, dx = 0, \quad j = 1, 2, \ldots,$$

where $U(x) \in L^2$. Then $UG^{1/2} \in L$ and since the sequence $\theta_j$ is closed in $L$ relation (13) implies $UG^{1/2} = 0$ almost everywhere. Thus $U = UG^{1/2}G^{-1/2} = 0$ almost everywhere. This proves that the sequence $\theta_j G^{1/2}$ is closed in $L^2$.

Closure in $L^2$ implies completeness in $L^2$ so it is possible to make $E$ arbitrarily small. This may be accomplished by choosing, for each $m$, the set $(\rho_1, \ldots, \rho_m)$ which minimizes $E$. As is well known this optimal choice is determined by the orthogonality equations

$$0 = \int \left( H - \sum_{k=1}^{m} \rho_k \theta_k \right) \theta_j^* G \, dx, \quad j = 1, 2, \ldots, m.$$

Since $HG = F$ it is apparent that these are precisely equations (10) of Theorem 1.

By the Schwarz inequality

$$\left( \int |H - \sum_{k=1}^{m} \rho_k \theta_k| \, dx \right)^2 \leq E \int |G^{-1}| \, dx.$$

The left side approaches zero if $E$ approaches zero. This shows that the optimal choice of $\rho_j$ leads to the satisfaction of condition (3) of Problem 2 and completes the proof.
Theorem 2. Suppose $F(x) \in L^2$ and $1/F(x) \in L^2$. Let

\begin{equation}
\beta_{jk\alpha} = \int_R \theta^\alpha_j(x) \theta_k(x) \theta^\alpha_k(x) \theta_\alpha(x) \, dx,
\end{equation}

\begin{equation}
g_{jk} = \sum_1^\infty \sum_1^\infty \beta_{jk\alpha} f^\alpha_i f_i.
\end{equation}

Then the equations

\begin{equation}
f_j = \sum_1^m g_{jk} p_k, \quad j = 1, 2, \ldots, m,
\end{equation}

may be solved for $p_1, p_2, \ldots, p_m$. For $m = 1, 2, \ldots$ these solutions form a set of expansion coefficients of the function $H(x) = 1/F^*(x)$.

Proof. Define $G(x) = F(x)F^*(x)$ then $H(x) = F(x)/G(x)$. Then $G(x) \geq 0$, $G(x) \in L$, $(G(x))^{-1} \in L$, and $H^2(x)G(x) \in L$. Thus the conditions of Theorem 1 are satisfied. Substituting $G = FF^*$ in relation (6) for the Fourier matrix gives

\begin{equation}
g_{jk} = \int \theta^\alpha_j(x) \theta_k(x) F^* F \, dx.
\end{equation}

The series $\sum_1^n f_i \theta^\alpha_i$ and $\sum_1^n f_i \theta_i$ converge in $L^2$ mean to $F^*$ and $F$. Therefore it is permissible to substitute these series in (19) and to integrate termwise. This justifies (17); the proof then follows from Theorem 1.

Theorem 3. Let $G(x) \geq 0$, $G(x) \in L$, $1/G(x) \in L$, and $F^2(x)/G(x) \in L$ in the interval $0 \leq x \leq 1$. The Fourier coefficients are now defined as

\begin{equation}
f_j = \int_0^1 F(x) \exp(-2\pi i j x) \, dx, \quad j = 0, \pm 1, \ldots.
\end{equation}

Then the equations

\begin{equation}
f_j = \sum_{-m}^m g_{j-k} p_k, \quad j = -m, \ldots, m,
\end{equation}

may be solved for $p_{-m}, \ldots, p_m$ and

\begin{equation}
\int_0^1 \left| F(x)/G(x) - \sum_{-m}^m \rho_k \exp(2\pi i k x) \right| \, dx \to 0 \quad \text{as} \quad m \to \infty.
\end{equation}

Proof. Theorem 1 is specialized with the sequence $\theta(x)$ being the sequence $\exp(2\pi i j x)$ suitably reordered. It is then seen that

\begin{equation}
g_{jk} = g_{j-k}
\end{equation}
and thus equation (10) becomes equation (21). The proof of relation (22) then follows from Theorem 1.

**Theorem 4.** Suppose $F(x) \in L^2$ and $1/F(x) \in L^2$ on the interval $0 \leq x \leq 1$. Let $f_j$ denote the Fourier coefficients of $F(x)$ relative to the orthonormal sequence $\exp(2\pi i j x)$. Let

\begin{equation}
\tag{24}
g_j = \sum_{-\infty}^{\infty} f_{j+r}^*.
\end{equation}

Then the equations

\begin{equation}
\tag{25}
f_{-j}^* = \sum_{-m}^{m} g_{j-k} \alpha_k
\end{equation}

may be solved for $\alpha_{-m}, \ldots, \alpha_m$ and

\begin{equation}
\tag{26}
\int_{0}^{1} \left| (F(x))^{-1} - \sum_{-m}^{m} \alpha_k \exp(2\pi i k x) \right| dx \to 0 \text{ as } m \to \infty.
\end{equation}

**Proof.** First Theorem 2 is applied to obtain $H = 1/F^*$. It is seen from (16) that $\beta_{j+r+s} = 1$ if $s = j - k + r$ and $\beta_{j+r+s} = 0$ otherwise. Thus (17) gives

\begin{equation}
\tag{27}
g_{j+k} = g_{j-k} = \sum_{-\infty}^{\infty} f_{j-k+r}^*.
\end{equation}

Then (18) is

\begin{equation}
f_j = \sum_{-m}^{m} g_{j-k} \rho_k.
\end{equation}

Taking the complex conjugate of this relation gives

\begin{equation}
f_{-j}^* = \sum_{-m}^{m} g_{j-k}^* \rho_k^*.
\end{equation}

But $g_{j-k}^* = g_k$ and $\rho_{j-k}^* = \alpha_k$ so this proves (25).

The questions studied in this note originated from the need for an algorithm which would give the reciprocal of a Laurent series. (In this connection it is worth noting that there is a simple algorithm for the reciprocal of a Taylor series.) An algorithm for the reciprocal of a Laurent series is stated in the following Corollary of Theorem 4.

**Corollary.** Consider the Laurent series

\begin{equation}
\tag{28}
S(z) = \sum_{-\infty}^{\infty} a_n z^n
\end{equation}

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which is assumed to converge when the complex variable $z$ is in the region $D$ defined by the relation $r_1 < |z| < r_2$. If $S(z)$ does not vanish in $D$ it follows, of course, that the reciprocal $T(z) = 1/S(z)$ is a Laurent series in $D$. Thus

\begin{equation}
T(z) = \sum_{-\infty}^{\infty} c_n z^n.
\end{equation}

Let coefficients $b_j$ be defined as

\begin{equation}
b_j = \sum_{-\infty}^{\infty} a_{j+r} a_r^*.
\end{equation}

Then the equations

\begin{equation}
a_j^* = \sum_{-m}^{m} b_{j-k} C_k^{(m)}, \quad j = -m, \ldots, m,
\end{equation}

may be solved for $C_{-m}^{(m)}, \ldots, C_m^{(m)}$. Let

\begin{equation}
T_m(z) = \sum_{-m}^{m} C_k^{(m)} z^k.
\end{equation}

Then as $m \to \infty$ the sequence $T_m(z)$ converges uniformly to $T(z)$ in any closed region contained in $D$. Moreover, $C_k^{(m)} \to c_k$ as $m \to \infty$.

PROOF. The function $F(z) = S(re^{2\pi i z})$ satisfies the condition of Theorem 4 for any value of $r$ in the range $r_1 < r < r_2$. The Fourier coefficients of $F$ are given by $f_j = r^j a_j$. Defining $b_j$ and $C_k^{(m)}$ by the relations $g_j = r^j b_j$ and $\alpha_j = r^j C_k^{(m)}$ it results that relations (24) and (25) yield relations (29) and (30). It follows that $T_m$ converges in mean to $T$ for $|z| = r$. Likewise $T_m$ converges in mean for $|z| = r'$ where $r_1 < r < r' < r_2$. Then by a standard theorem from complex function theory it follows that $T_m$ converges uniformly to $T$ for $r < |z| < r'$. This is seen to complete the proof of the Corollary.

REFERENCE


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