lows from our lemma by much the same sort of argument that pro-
duced our basic theorem.

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UNCOUNTABLY MANY NONISOMORPHIC NILPOTENT LIE ALGEBRAS

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Throughout this note, L denotes a Lie algebra over the real number field R. We shall define $L^i$ and $L_i$ inductively. $L = L^0 = L_0$, $L^i = [L^{i-1}, L^{i-1}]$, and $L_i = [L, L_{i-1}]$ for all integers $i \geq 1$. Thus, $L^i$ is the space of all finite sums $\sum [x, y]$, $x, y \in L^{i-1}$. Similarly, $L_i$ is the space of all finite sums $\sum [x, y]$, $x \in L$ and $y \in L_{i-1}$. If $L^r = 0$ and $L^{r-1} \neq 0$, $L$ is said to be solvable of index $r$. If $L_i = 0$ and $L_{i-1} \neq 0$, $L$ is said to be nilpotent of length $t$.

DEFINITION. Let $F$ be a subfield of $R$. A Lie algebra $L$ over $R$ is said to be an $F$-algebra if its structure constants with respect to some basis of $L$ lie in $F$.

Malcev [1] showed that for each integer $n \geq 16$ there is a nilpotent Lie algebra of length 2 and dimension $n$ which is not a rational algebra. The purpose of this note is to prove the following theorem which contains an improvement of Malcev's result:

THEOREM. There exist uncountably many nonisomorphic nilpotent Lie algebras of length 2 for any given dimension $N \geq 10$.

Following from the theorem we can easily show:

COROLLARY 1. There exist uncountably many solvable not nilpotent Lie algebras of index 3 for any given dimension $M \geq 11$.

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1 This is a portion of my thesis submitted to the University of Michigan. I am
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Let $E$ be a subfield of $R$, let $m$, $n$ be two natural numbers, and let $c_{jk}$, $i=1, 2, \ldots, n$, $j, k=1, 2, \ldots, m$, be real numbers such that $c_{jk} = -c_{kj}$. Also let $L$ be a Lie algebra over $R$ defined by a basis $(x_1, \ldots, x_m, y_1, \ldots, y_n)$ with products $[x_j, x_k] = \sum_{l=1}^m c_{jl} y_l$ for $j, k=1, 2, \ldots, m$, and all other products zero, so that $L$ is nilpotent of length $\leq 2$.

**Lemma.** If the numbers $c_{jk}$, $1 \leq i \leq n$, $1 \leq j < k \leq m$, are algebraically independent over $E$, and if $(n/2)(m^2 - m) > m^2 + n^2$, then $L$ is not an $E$-algebra.

**Proof.** We first note that $(n/2)(m^2 - m) > m^2 + n^2$ implies $(1/2)(m^2 - m) > n$. Any $n$ different elements $[x_j, x_k], j < k$, of $L_1$ are linearly independent, since the determinant formed by $c_{jk}$ involved cannot be zero by the algebraic independence of all $c_{jk}$. It follows that $L_1$ is generated by $y_1, y_2, \ldots, y_n$, denoted by $L_1 = \{(y_1, y_2, \ldots, y_n)\}$, since for any $x \in L_1$ there exist $u_i, v_i$ such that $x = \sum_i [u_i, v_i]$ which is a linear combination of $y_i$’s. We also note that the center of $L$ is exactly $L_1$; let $x$ be any element of the center, then $x = \sum_{j=1}^m a_j x_j + \sum_{i=1}^n b_i y_i$, and $0 = [x, x_k] = \sum_{j=1}^m a_j \sum_{i=1}^n c_{jk} y_i$ for $k = 1, 2, \ldots, m$, By linear independence of the $\{y_i\}$, we have $\sum_{j=1}^m a_j c_{jk} = 0$ for $i = 1, 2, \ldots, n$, and $k = 1, 2, \ldots, m$, i.e., there are $n \cdot m$ equations and $m$ unknowns. By the algebraic independence of all $c_{jk}$, the rank of the coefficient matrix in the system of homogeneous equations is equal to $m$. Hence, we have $a_1 = a_2 = \cdots = a_m = 0$. Consequently, $x = \sum_{i=1}^n b_i y_i$ and the center is $L_1$, and $L$ is of length 2.

Suppose now that $L$ is an $E$-algebra with basis $(z_1, \ldots, z_m, z_{m+1}, \ldots, z_{m+n})$ and structure constants $d_{ij}^k$, $1 \leq i, j, k \leq m+n$, lying in $E$. We can assume that $(z_1, \ldots, z_m)$ are independent modulo $L_1$, i.e., they span a complement $C$ of $L_1$ in $L$. We can write $z_{m+i} = v_i + t_i$ with $v_i \in C$ and $t_i \in L_1$ for $i = 1, 2, \ldots, n$. Clearly, $(z_1, z_2, \ldots, z_m, t_1, \ldots, t_n)$ is still a basis for $L$. We have

$$[z_i, z_j] = \sum_{r=1}^m d_{ij}^r z_r + \sum_{s=m+1}^{m+n} d_{ij}^s z_{s+m} + \sum_{s=m+1}^{m+n} d_{ij}^s z_{s-m},$$

for $1 \leq i, j \leq m$. But since $[z_i, z_j] \subseteq L_1$, the first two sums, which are in $C$, must be zero. Hence we have

$$[z_i, z_j] = \sum_{r=1}^n d_{ij}^{m+r} t_i, \quad \text{for} \quad i, j = 1, 2, \ldots, m.$$

These equations describe the multiplication in $L$ in the basis $(z_1, \ldots, z_m, t_1, \ldots, t_n)$; the structure constants are part of the structure constants for the basis $(z_1, \ldots, z_m, z_{m+1}, \ldots, z_{m+n})$. 

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We note that \(((x_1, \cdots, x_m))\) also forms a complement of \(L_1\) in \(L\) say \(C\). It follows that we can replace each \(z_i\) by an element \(s_i\) such that \(s_i - z_i \in L_1\) and \(s_i \in C\). Since \(L_1\) is the center of \(L\), the structure constants for the basis \((z_1, \cdots, z_m, t_1, \cdots, t_n)\) are the same as for the basis \((s_1, \cdots, s_m, t_1, \cdots, t_n)\) above.

The set of vectors \(\{s_1, \cdots, s_m\}\) is of course a basis for \(C\), and, therefore, we have \(s_i = \sum_{p=1}^{m} a_{ip} x_p, i = 1, \cdots, m\), where \(A = (a_{ip})\) is a nonsingular matrix. Similarly, \(t_g = \sum_{r=1}^{n} b_{gr} y_r, g = 1, \cdots, n\), with nonsingular matrix \(B = (b_{gr})\). Substituting into \([s_i, s_j] = \sum_{u=1}^{n} d_{ij} t_u, 1 \leq i, j \leq m\), we obtain, by linear independence,

\[
\sum_p \sum_r a_{ip} a_{j\delta} c_{pr} = \sum_u d_{ij} b_{ur},
\]

for fixed \(i, j\), and \(r\).

This means, with \(a_{ij} = (A^{-1})_{ij}\), that

\[
c_{pr} = \sum_{i} \sum_{j} \sum_{u} d_{ij} b_{ur} a_{pi} a_{qj}.
\]

These equations imply that the \(c_{pr}\) lie in the field \(E(a_{ip}, b_{ur})\), but this field has degree of transcendency over \(E\) at most \(m^2 + n^2\) which is a contradiction. Hence, \(L\) is not an \(E\)-algebra.

The smallest dimension to which this applies is 10 with \(m = 6\) and \(n = 4\). In fact, the lemma applies to any dimension \(N \geq 10\), because when \(N \geq 10, N^2 - 10N + 8 > 0\) holds, implying that \(n(m^2 - m)/2 > m^2 + n^2\) holds for \(n = 4\) and \(m = N - 4\).

Now the proof of the theorem: It is well known that there exists a set, \(S\), of uncountably many real numbers which are algebraically independent over the rational number field \(Q\). With \(n = 4\) and \(m = N - 4\), we divide \(S\) into disjoint subsets \(\{c_{ik}\}\) (the Greek index distinguishes the various subsets), each of which is restricted to values of \(j\) and \(k\) such that \(j < k\) and \(c_{jk} = -c_{kj}\). Write \((L)_\alpha = ((x_1, \cdots, x_m, y_1, \cdots, y_4))\) with products \([x_j, x_k] = \sum_{i=1}^{4} c_{jk} y_i, j, k = 1, 2, \cdots, m\), and all other products zero. There are still uncountably many such subsets \(\{c_{jk}\}\) since each \(\{c_{jk}\}\) is finite. Consequently, there are uncountably many such Lie algebras \((L)_\alpha\). We claim that any two \((L)_\alpha\) and \((L)_{\alpha'}\) are nonisomorphic. Since \(c_{jk}\) are algebraically independent over \(Q(\{c_{jk}\}_{\alpha'})\), apply the lemma with \(E = Q(\{c_{jk}\}_{\alpha'})\).

Now the proof of Corollary 1: In the proof of the theorem we have seen that for each \(\alpha\), \((L)_{\alpha} = ((x_1, \cdots, x_{N-4}, y_1, \cdots, y_4))\), with \([x_j, x_k] = \sum_{i=1}^{4} c_{jk} y_i\) for \(j, k = 1, \cdots, N - 4\) where \(N \geq 10\), and all
other products zero, is a nilpotent Lie algebra of dimension $N$ and length 2. Let $(L')_a = ((x_1, \cdots, x_{N-4}, y_1, \cdots, y_4, z_a))$ where the multiplications of $x_j$'s and $y_i$'s are defined as same as in $L$ and $[z_a, x_j] = x_j$, $[z_a, y_i] = 2y_i$, for $j = 1, \cdots, N-4; i = 1, \cdots, 4$ and $N \geq 10$. Then clearly, $(L)_a$ is a solvable not nilpotent Lie algebra of dimension $M = N+1 \geq 11$. Any two such Lie algebra $(L')_a$ and $(L')_a'$ are clearly nonisomorphic because by the theorem their commutators are non-isomorphic.

**Corollary 2.** There are uncountably many nonisomorphic non-rational nilpotent Lie algebras of length 2 for any given dimension $N \geq 10$.

**Remarks.** We note that the uncountability of nonisomorphic solvable Lie algebras is quite different from the case of semisimple Lie algebras where in each dimension there are only a finite number of nonisomorphic ones.

**Bibliography**