1. Introduction. Using the notation and terminology of Ahlfors-Sario [3], we shall denote by $\Gamma_h$ the Hilbert space of square integrable harmonic differentials on a Riemann surface $W$. Let $\gamma$ be a 1-chain on $W$ and let $\gamma$ be the element in $\Gamma_h$ with the property, $(\omega, \gamma) = \int \omega$ for all $\omega \in \Gamma_h$. We refer to $\gamma$ as a reproducing kernel for periods for $\Gamma_h$. Let $\xi$ be a point of $W$ and $z = x + iy$ be a local parameter near $\xi$ such that $z(\xi) = 0$. A $\Gamma_h$-kernel for $n$th derivatives at $\xi$ is a differential $\gamma \in \Gamma_h$ which satisfies $(\omega, \gamma) = (\partial^n/\partial x^n)u(0)$ for all $\omega \in \Gamma_h$ where $\omega = du(z)$ near $\xi$. This kernel is uniquely determined by $\xi$, the uniformizer $z$, and the positive integer $n$. If in the above definitions we replace $\Gamma_h$ by one of its subspaces then we shall refer to the corresponding kernels as reproducing kernels for that subspace. It is easily seen that such a kernel is the orthogonal projection of a $\Gamma_h$-kernel.

The existence of these kernels is well known (see [1; 3]). The possibility of expressing them explicitly in terms of principal harmonic functions (the existence of which has been proved constructively by Sario [5]) was first investigated by Weill [6]. In this paper the kernels for the spaces $\Gamma_h$, $\Gamma_{h0}$, $\Gamma_{he}$, $\Gamma_{he}$, $\Gamma_{hm}$, $\Gamma_{h0}$, $\Gamma_{h0} \cap \Gamma_{h0}$, $\Gamma_{h0} \cap \Gamma_{h0}$, $\Gamma_{h0}$, $\Gamma_{h0}$, $\Gamma_{h0}$, and some spaces associated with a regular partition of the ideal boundary of the surface are found in terms of principal functions.

§§9 and 10 deal with some applications of these results. A characterization of Ahlfors' class of quasi-rational functions is given in terms of principal functions. A Riemann-Roch type theorem is proved for arbitrary Riemann surfaces which reduces to a theorem of Royden [4] in case the surface is of class $O_{KD}$.

2. Reproducing kernels. We shall apply the general method of normal operators [3] to the following situation. Let $h$ be a harmonic function defined in a boundary neighborhood of a Riemann surface $W'$ and assume that $h$ has zero flux along the ideal boundary of $W'$. The principal operator $L_0$ maps $h$ into a function $\varphi_h$ which is harmonic on $W'$ and is characterized up to an additive constant by the

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982
property that the normal derivative of \( p_0 - h \) vanishes identically along the ideal boundary of \( W' \). When \( W' \) is the complement of a finite subset \( \{ \xi_1, \ldots, \xi_n \} \) of a Riemann surface \( W \) then \( p_0 - h \) has a removable singularity at each \( \xi_i \). We say that it has \( L_0 \) behavior at the ideal boundary of \( W \). Let \( P \) be a regular partition of this ideal boundary [3, p. 165]. The \( (P)L_1 \) principal operator associates to \( h \) a function \( p_{PL} \), harmonic on \( W' \) with the singularity \( h \) at each \( \xi_i \). Furthermore, \( p_{PL} - h \) is constant on each ideal boundary component of \( W \) determined by \( P \) and has zero flux over each \( P \)-dividing cycle.

Let \( I \) and \( Q \) denote, respectively, the identity and the canonical partitions of the ideal boundary of \( W \) in the sense of [3]. In terms of a uniformizer \( z \) near \( \xi \in W \) such that \( s(\xi) = 0 \), we let \( s = \text{Re } z^{-n} \) and \( t = \text{Im } z^{-n} \) for a positive integer \( n \), and define \( s = t = 0 \) outside of a compact subset of \( W \). The differential \( dp_{PL} + dp_{PL}^* \) has a removable singularity at \( \xi \) and is an element of \( \Gamma \).

**Theorem 1.** The differential \( dp_{PL} + dp_{PL}^* \) reproduces for the space \( \Gamma \).
Explicitly, we have
\[
(\omega, dp_{PL} + dp_{PL}^*) = -\frac{2\pi}{(n-1)!} \frac{\partial^n}{\partial s^n} u(0)
\]
where \( \omega = du(z) \) near \( \xi \).

3. Before proving the theorem we shall first establish a lemma. A subregion \( \Omega \subset W \) is called canonical if it is relatively compact, its boundary consists of a finite number of analytic Jordan curves, and each component of \( W - \Omega \) is noncompact and bounded by a single contour. For a first order differential \( \omega \) let \( f_\omega \) indicate the limit as \( \Omega - \Omega \) of integrals taken along the boundaries of exhausting canonical subregions \( \Omega \).

**Lemma.** Let \( p \) be the \( (I)L_1 \) principal function for any singularity. Then \( \int_{\partial \Omega} f_\omega \) is 0 for all \( \omega \in \Gamma \).

If \( \Omega \subset W \) is a canonical subregion with boundary \( \partial \Omega \) then \( \int_{\partial \Omega} f_\omega \) is the \( (I)L_1 \) principal function for the surface \( \Omega \) which has the same singularity as that of \( p \). By Stokes' theorem the right side of this equation is the inner product \( (dp - dp_\Omega, \omega^*)_\omega \) taken over the region \( \Omega \). By Schwarz's inequality this inner product is dominated by \( \|dp - dp_\Omega\|_\omega \cdot \|\omega\|_\omega \). We therefore obtain \( \int_{\partial \Omega} f_\omega \) is 0 for all \( \omega \). The uniform convergence \( p_\Omega \to p \) on compact subsets of \( W \) implies that \( \|dp - dp_\Omega\|_\omega \to 0 \).

For the proof of Theorem 1 we may assume that \( \omega \) is real. Let \( \Delta \) be a relatively compact neighborhood of \( \xi \) which is mapped by \( z \) onto a
disk \{ |z| < r \} and let \( \partial \Delta = \alpha \). In order to simplify the notation we shall temporarily denote the functions \( p_{r_1} \) and \( q_{r_1} \) by \( p \) and \( q \) respectively. In \( \Delta \) let \( \omega = du(z) \) and \( \rho = \text{Re } z^{-n} + v(z) \) where \( u \) and \( v \) are harmonic in \( \{ |z| < r \} \). We then obtain

\[
(\omega, dp + dq^*) = \int \int_{W} \omega(dp^* - dq) = \int \int_{W-A} \omega dp^* - \omega dq + \int \int_{\Delta} \omega(dp^* - dq).
\]

The integral over \( W-A \) is the limit as \( \Omega \to W \) of integrals over \( \Omega - \Delta \). We may apply Stokes' theorem to these integrals and, after simplification, we find

\[
(\omega, dp + dq^*) = \int_{\beta} (p \omega^* + q \omega) - \int_{\alpha} (p \omega^* + p^* \omega).
\]

The integral along \( \beta \) vanishes by the lemma. Hence

\[
(\omega, dp + dq^*) = - \int_{|z|=r} (\text{Re } z^{-n} + v)du^* + (\text{Im } z^{-n} + v^*)du
\]

\[
= \text{Re} \int (u^* - iu)d(z^{-n}) - \int vdu^* - uvd^*
\]

\[
= \text{Re} \, ni \int \frac{u + iu^*}{z^{n+1}} \, dz
\]

\[
= \frac{2\pi}{(n-1)!} \frac{\partial^n}{\partial x^n} u(0).
\]

4. We now seek an expression for the reproducing differential for periods. Let \( \Delta \) be a parametric disk on \( W \) and \( \gamma \) a 1-simplex contained in \( \Delta \). In terms of the parameter \( z \) which maps \( \Delta \) onto the unit disk we define a singularity function \( \sigma = \log \left| (z - \zeta_2)/(z - \zeta_1) \right| \), where \( \partial \gamma = \zeta_2 - \zeta_1 \). We set \( \sigma = 0 \) outside of a compact set. The corresponding \((P)L_1\) principal function is \( p_{Pr} \). Let \( \tau \) be the singularity function \( \tau = \arg (z - \zeta_2)/(z - \zeta_1) \) in \( \Delta - \gamma \) and \( \tau = 0 \) near the ideal boundary of \( W \). On the surface \( W - \gamma \) we choose the normal operator which is composed of \((P)L_1\) for a boundary neighborhood of \( W \) and of the Dirichlet operator for \( \Delta - \gamma \). This Dirichlet operator maps a continuous function on \( \partial \Delta \) into the restriction to \( \Delta - \gamma \) of the harmonic function in \( \Delta \) with these boundary values. The direct sum of these operators yields a function \( p_{Pr} \), harmonic on \( W - \gamma \). The differential \( dp_{Pr} \) can be ex-
tended harmonically to all of $W - \{\zeta_1, \zeta_2\}$. We shall continue to denote the extension by $dp_{P\tau}$, even though it is not exact. If $\gamma$ is an arbitrary 1-chain it is homologous to a finite sum $\sum n_i \gamma_i$, where each $\gamma_i$ is a 1-simplex contained in a parametric disk and each $n_i$ is an integer. We extend the definitions of $dp_{P\sigma}$ and $dp_{P\tau}$ to arbitrary $\gamma$ by letting $dp_{P\sigma} = \sum n_i dp_{P\gamma_i}$ and similarly for $dp_{P\tau}$. That these differentials are well defined will follow from Theorem 2.

**Theorem 2.** The differential $dp_{P\sigma} + dp_{P\tau}^*$ reproduces for $\Gamma_h$. Specifically, if $\sigma$ and $\tau$ correspond to a 1-chain $\gamma$ then

$$(\omega, dp_{P\sigma} + dp_{P\tau}^*) = 2\pi \int_{\gamma} \omega$$

for all $\omega \in \Gamma_h$.

Because of the linearity it suffices to prove Theorem 2 for the case that $\gamma$ is a 1-simplex contained in a parametric disk $\Delta: \{|z| < 1\}$ and $\partial \gamma = \zeta_2 - \zeta_1$. We shall shorten the notation and write $p$ and $q$ for $p_{P\sigma}$ and $p_{P\tau}$ respectively. As in the proof of Theorem 1 it can be shown that

$$(\omega, dp + dq^*) = -\int_{\partial \Delta} (p\omega^* + p^*\omega),$$

where $\partial \Delta = \alpha$. In $\Delta$ let $\omega = du$ and $p = \log |(z - \zeta_2)/(z - \zeta_1)| + v$, where $u$ and $v$ are harmonic. Let $\alpha_1$ and $\alpha_2$ be disjoint circles in $\Delta$ with centers at $z(\zeta_1)$ and $z(\zeta_2)$ respectively. After applying Green's formula one obtains

$$(\omega, dp + dq^*) = \int_{\alpha_2} ud \log |z - \zeta_2|^* - \int_{\alpha_1} ud \log |z - \zeta_1|^*.$$  

By the mean value formula the last two integrals reduce to $2\pi(u(\zeta_2) - u(\zeta_1))$. We have shown that $(\omega, dp + dq^*) = 2\pi \int_{\partial \Delta} du = 2\pi \int_{\alpha} \omega$. This completes the proof of Theorem 2.

5. For a regular partition $P$ of the ideal boundary of $W$ the spaces $(P)\Gamma_{hm}$ and $(P)\Gamma_{he}$ were introduced in [3], and a proof of the decomposition $\Gamma_h = (P)\Gamma_{hm} + (P)\Gamma_{he}$ was sketched. It follows that $\Gamma_h = \Gamma_{he}^* + (P)\Gamma_{hm}^* + (P)\Gamma_{he}^* \cap \Gamma_{he}^*$ is an orthogonal direct sum. Consider the identity

$$(1) \quad -(dp_{P\sigma} + dp_{P\tau}^*) = (dp_{P\sigma} - dp_{P\tau})^* + (dp_{P\tau} - dp_{P\sigma}) - (dp_{P\sigma}^* + dp_{P\tau}^*),$$

where the zero subscript refers to the $L_0$ principal function. The first term on the right side of (1) is evidently in $\Gamma_{he}^*$. The second term is
the limit as $\Omega \rightarrow W$ of $dp_{P+\Omega} - dp_{P\Omega}$, where the subscript $\Omega$ indicates a principal function for the subsurface $\Omega$. These approximating differentials belong to the space $(P)\Gamma_{hm}(\Omega)$ of harmonic differentials $du$ on $\Omega$ with $u$ constant on each set of contours of the boundary of $\Omega$ which belong to the same part in the induced partition. We may conclude that the limit differential is in $(P)\Gamma_{hm}$. The last term on the right side of (1) is a limit of elements from $\Gamma_{ho}(\Omega)$. By Theorem V.14C of [3], we see that it is in $\Gamma_{ho}$. Its conjugate has vanishing periods along any cycle which is dividing relative to the partition $P$. Hence it is in $(P)\Gamma_{h}^* \cap \Gamma_{ho}$. Thus (1) represents the reproducing kernel for $\Gamma_{h}$, except for a multiplicative constant, as the sum of the kernels for $\Gamma_{mh}$, $(P)\Gamma_{hm}$, and $(P)\Gamma_{h}^* \cap \Gamma_{ho}$. The reasoning is also valid for the kernels corresponding to a 1-chain $\gamma$. This proves the following two corollaries.

**Corollary 1.** The differentials

$$\frac{(n - 1)!}{2\pi} (dp_{P+} - dp_{P\Omega})$$

and

$$\frac{1}{2\pi} (dp_{P\Omega} - dp_{P+})$$

are the reproducing kernels for the space $(P)\Gamma_{hm}$.

**Corollary 2.** The differentials

$$\frac{-(n - 1)!}{2\pi} (dp_{P+} + dp_{P\Omega}^*)$$

and

$$\frac{1}{2\pi} (dp_{P\Omega} + dp_{P+}^*)$$

are the reproducing kernels for $(P)\Gamma_{h}^* \cap \Gamma_{ho}$.

Since $(L)\Gamma_{h}^* \cap \Gamma_{ho} = \Gamma_{ho}$, Corollary 2 contains the following special case.

**Corollary 3.** The differentials

$$\frac{-(n - 1)!}{2\pi} (dp_{P+} + dp_{P\Omega}^*)$$
and
\[ \frac{1}{2\pi} (d\rho_{\omega} + d\rho_{\mu}) \]
are the reproducing kernels for $\Gamma_0$.

Remarks. In order to indicate explicitly the relation between the
differentials in (1) and the classical domain functions we denote by
$N(z, \xi)$ and $G(z, \xi)$ the Neumanns' and Green's functions for $W$. In
case $W$ is parabolic let $G(z, \xi) = G(z, \xi_0)$ be the fundamental
potential [4]. In terms of the complex notation
\[ d' = \frac{\partial}{\partial z}, \quad d'' = \frac{\partial}{\partial \bar{z}}, \quad d^c = i(d'' - d') \]
(so that $d^c f = d f^*$), we have
\[ dp_{\omega} + dp_{\mu} = \frac{4}{(n-1)!} \operatorname{Re} \frac{\partial^n G}{\partial \bar{z}^n}, \]
\[ (dp_{\omega} - dp_{\mu})^* = \frac{2}{(n-1)!} \operatorname{Im} \frac{\partial^n}{\partial \bar{z}^n} (N - G), \]
\[ dp_{\omega} + dp_{\mu} = 4 \operatorname{Re} \int_{\gamma} d' \, d^c G. \]
Theorems 1 and 2 can also be derived from the fact that $d''(\partial G/\partial \xi)$
reproduces for $\Gamma$. If $W$ is the interior of a compact bordered surface
then by classical theory there is a potential $G_P(z, \xi_0; \xi, \xi_0)$ which as a
function of $z$ is constant and has zero flux along each $P$-boundary
component. In this case we have
\[ dp_{\omega} - dp_{\mu} = \frac{2}{(n-1)!} \operatorname{Re} \frac{\partial^n}{\partial \bar{z}^n} (G_P - G), \]
\[ dp_{\omega}^* + dp_{\mu} = \frac{2}{(n-1)!} \operatorname{Re} (dG_P - id^c N). \]

6. The space $\Gamma_\mu$ has the orthogonal decomposition
\[ \Gamma_\mu = (P)\Gamma_{hm} + \Gamma_0^* + (P)\Gamma_{ke}^* \cap \Gamma_{he}. \]
Since we are in possession of reproducing kernels for $(P)\Gamma_{hm}$ and $\Gamma_0^*$
(see the proof of Corollary 1) we obtain the next result immediately.

Corollary 4. The differentials
\[ \frac{(n-1)!}{2\pi} (dp_{\omega} - dp_{\mu}) \]
and
\[ \frac{1}{2\pi} (d\phi_{\rho e} - d\phi_{\rho o}) \]
are the reproducing kernels for \((P)\Gamma_{h e}^* \cap \Gamma_{h e}\).

The reproducing kernel for \(\Gamma_{h e}\) may be derived from Corollary 4.

**Corollary 5.** The differentials
\[ \frac{(n - 1)!}{2\pi} (d\phi_{\rho e} - d\phi_{\rho o}) \]
and
\[ \frac{1}{2\pi} (d\phi_{\rho e} - d\phi_{\rho o}) \]
are the reproducing kernels for \(\Gamma_{h e}\).

7. The kernel for \((P)\Gamma_{h e}^*\) can be found from the identity
\[ -(d\rho_{1 e} + d\rho_{1 e}^*) = (d\rho_{1 e} - d\rho_{1 e}^*) - (d\rho_{1 e}^* + d\rho_{1 e}^*) \]
and the orthogonal decomposition \(\Gamma_h = (P)\Gamma_{h e}^* + (P)\Gamma_{h e}\).

**Corollary 6.** The differentials
\[ \frac{-(n - 1)!}{2\pi} (d\rho_{1 e} + d\rho_{1 e}^*) \]
and
\[ \frac{1}{2\pi} (d\rho_{1 e} + d\rho_{1 e}^*) \]
are the reproducing kernels for \((P)\Gamma_{h e}^*\).

8. From the orthogonal direct sum,
\[ \Gamma_h = (P)\Gamma_{h m} + (P)\Gamma_{h m}^* + (P)\Gamma_{ae} + (P)\Gamma_{ae}, \]
we can project the \(\Gamma_h\)-kernels into \((P)\Gamma_{ae}\).

**Corollary 7.** The differentials
\[ \frac{-(n - 1)!}{4\pi} (d\rho_{p e} + d\rho_{p e}^* + i(d\rho_{p e}^* - d\rho_{p e})) \]
are the reproducing kernels for \((P)\Gamma_{ae}n\).

The kernels for \(\Gamma_n\) and \(\Gamma_{ae}\) may be found immediately from Corollary 7.

**Remarks.** It is an open question whether results analogous to those above can be obtained for the subspaces \(\Gamma_S\) and \(\Gamma_{as}\). Investigations in this area might lead to new normal operators.

9. The results of §§1–8 indicate a connection between principal functions and Ahlfors' generalization of Abel's theorem [2, 3]. Recall that a differential \(\omega\) on \(W\), harmonic except for harmonic poles, is called distinguished if

(i) \(\omega^*\) is semieexact outside of some compact subset of \(W\),
(ii) there exist differentials \(\omega_{hm} \in \Gamma_{hm}\) and \(\omega_{e0} \in \Gamma_{e0}\) such that \(\omega = \omega_{hm} + \omega_{e0}\) in a boundary neighborhood of \(W\).

**Lemma.** Let \(\gamma\) be a \(1\)-chain on \(W\) and \(\delta\) be a \(1\)-cycle. Let \(dp_{Ps}\) and \(dp_{Pr}\) be differentials associated with \(\gamma\) (see §4). Then

\[
\int_{\delta} (dp_{Ps} + idp_{Pr}) = 2\pi i (\delta \times \gamma).
\]

**Proof.** Because (2) is linear in \(\gamma\) we may assume that \(\gamma\) is a simplex contained in a parametric disk \(\Delta: \{ |z| < 1 \}\). Since \(dp_{Ps} + idp_{Pr}\) has no periods along cycles in \(W - \Delta\) we may even assume \(\delta \subset \Delta\). Let \(\partial \gamma = \zeta_2 - \zeta_1\). Then \(dp_{Ps} + idp_{Pr} = d \log (z - \zeta_2)/(z - \zeta_1) + du\) in \(\Delta\), where \(u\) is harmonic. The proof reduces to establishing

\[
\int_{\delta} d \log \frac{z - \zeta_2}{z - \zeta_1} = 2\pi i (\delta \times \gamma).
\]

Since \(\delta \times \gamma\) is the number of times \(\gamma\) crosses \(\delta\) from left to right, (3) follows from the argument principle.

With the help of the above lemma it can be seen that to each distinguished differential \(\omega\) there corresponds a differential \(\lambda(P, \omega)\) with the singularities and periods of \(\omega\) and which, in a boundary neighborhood of \(W\), is the differential of a function whose real and imaginary parts have \((P)L_1\)-behavior.

**Theorem 3.** Let \(\omega\) be a distinguished differential. Then \(\lambda(Q, \omega) = \omega\).
Proof. Because of the uniqueness theorem for distinguished differentials it suffices to prove merely that \( \lambda(Q, \omega) \) satisfies conditions (i) and (ii) above. The first condition is obvious.

The differential \( \omega_{\alpha} = \lambda(Q, \omega) - \lambda(I, \omega) \) is in \( \Gamma_{h\omega} \). Let \( \Omega \) be a canonical subregion of \( W \) such that \( \omega \) is regular and exact in \( W - \Omega \). Choose another canonical region \( \Omega' \supset \Omega \). By standard techniques one can construct an exact differential \( \omega_{\alpha} \) in \( \Gamma_{1} \) with the property,

\[
\omega_{\alpha} = \begin{cases} 
0 & \text{in } \Omega, \\
\lambda(I, \omega) & \text{in } W - \Omega'.
\end{cases}
\]

That \( \omega_{\alpha} \in \Gamma_{\alpha} \) follows from the fact that an \((I)L_{1}\) principal function for \( W \) is the limit, uniformly on compacta, of \((I)L_{1}\) principal functions for bordered exhausting canonical subregions. The equality \( \lambda(Q, \omega) = \omega_{\alpha} + \omega_{\alpha} \) is valid in \( W - \Omega' \) and hence \( \lambda(Q, \omega) \) is distinguished.

From the above considerations one sees that a meromorphic function \( f \) is quasi-rational if and only if \( \log f \) has a single-valued branch outside of a compact subset of \( W \), the real and imaginary parts of which are \((Q)L_{1}\) principal functions.

Remark. There is a natural interpretation for “quasi-rational with respect to a regular partition \( P \).” The corresponding generalized Abel’s theorem is obtained by replacing \( \Gamma_{\alpha} \) by \((P)\Gamma_{\alpha}\).

10. Riemann-Roch type theorems. The classical theorem of Riemann-Roch has been extended to surfaces of class \( O_{KD} \) by Royden [4]. By the methods of this paper we obtain a similar theorem for an arbitrary Riemann surface.

Let \( D \) be a divisor on a Riemann surface \( W \) and let \( D = B - A \) where \( A \) and \( B \) are disjoint integral divisors. Suppose that \( A = \sum m_{j}A_{j} \) and \( B = \sum n_{k}B_{k} \), where the \( A_{j} \) and \( B_{k} \) are points of \( W \) and the \( m_{j} \) and \( n_{k} \) are positive integers. For each \( j \) and \( k \) let \( \Delta_{j} \) and \( \Delta'_{k} \) be parametric disks with centers at \( A_{j} \) and \( B_{k} \) respectively. We assume that these disks are mutually disjoint. Let \( \Delta = \bigcup \Delta_{j} \) and \( \Delta' = \bigcup \Delta'_{k} \).

Let \( S = s + it \) be a meromorphic function in \( \Delta' \) and a multiple of the divisor \( -B \). Define \( S = 0 \) in a boundary neighborhood of \( W \) and form the differential \( dF_{S} = dp_{P_{S}} + idp_{P_{S}} \). Let \( \mathfrak{B} \) denote the complex vector space consisting of all such \( dF_{S} \). The dimension of \( \mathfrak{B} \) is equal to the degree of \( B \), i.e., \( \dim \mathfrak{B} = \deg B = \sum n_{k} \).

In the decomposition \( dF_{S} = \phi_{S} + \psi_{S} \), where \( \psi_{S} = \frac{1}{2}(dF_{S} - idF_{S}) \), the differential \( \phi_{S} \) is meromorphic and \( \psi_{S} \) is in \((P)\Gamma_{\alpha} \). According to Corollary 7, \( \psi_{S} \) has certain reproducing properties. The following lemma is a generalization of that fact. For a divisor \( E \) let \((P)\Gamma_{\alpha} \) denote the vector space of meromorphic differentials which are mul-
tuples of $E$, square integrable near the ideal boundary of $W$, and which have vanishing periods along each $P$-dividing cycle outside of some compact set.

**Lemma.** Let $\alpha \in (P)\Gamma_{\ast e}[ - A ]$. Then

$$\tag{4} (\alpha, \psi_S)_{w-\Delta} = i \int_{\partial \Delta} S\alpha + i \int_{\partial \Delta} F_S \alpha. $$

The inner product is to be understood as a Cauchy limit. That is, it is the limit of inner products taken over $W-\Delta-\Delta'$ as the radii of the parametric disks of $\Delta'$ tend to zero. Note that

$$\tag{5} \int_{\partial \Delta} F_S \alpha = 0. $$

A modification of the proof of the lemma in §3 shows that $\int_{\partial \Delta} F_S \alpha \to 0$ as $\Omega \to W$. In $\Delta'$ we have $F_S = S + u$ where $S$ is meromorphic and $u$ is harmonic. When the disks about the $B_k$ shrink to points we obtain (4).

We shall make use of the following algebraic facts [3, p. 325]. Let $U$ and $V$ be vector spaces over a field $K$. A bilinear mapping $T: U \times V \to K$ is called a *pairing* of $U$ and $V$. The left kernel $U_0$ is the space $\{ u \in U : T(u, V) = 0 \}$, and the right kernel $V_0$ is the space $\{ v \in V : T(U, v) = 0 \}$. If one of the quotient spaces $U/U_0$ or $V/V_0$ is finite dimensional then there is an isomorphism $U/U_0 \cong V/V_0$.

Consider the pairing $T: \mathcal{G} \times (P)\Gamma_{\ast e}[ - A ] \to \mathbb{C}$ defined by $T(dF_S, \alpha) = \int_{\partial \Delta} S\alpha$.

Suppose $dF_S$ is in the left kernel of this pairing. From (4) we have $(\alpha, \psi_S)_{w-\Delta} = i \int_{\partial \Delta} F_S \alpha$ for all $\alpha \in (P)\Gamma_{\ast e}[ - A ]$. We may replace $\alpha$ by $\psi_S$ and conclude that $\| \psi_S \|_w = 0$. Hence $\psi_S = 0$ and $dF_S$ is meromorphic. We also find that

For appropriate choices of $\alpha$ we conclude from (5) that the additive constant in $F_S$ can be chosen so that $F_S$ is a multiple of the divisor $-D$. Conversely, if $dF_S$ is meromorphic then $\psi_S = 0$ and $T(dF_S, \alpha) = \int_{\partial \Delta} S\alpha = -\int_{\partial \Delta} F_S \alpha$. The differential $\alpha$ has a pole of order at most $m_i$ at $A_i$, and $F_S^{(k)} (A_i) = 0$ ($k = 1, \cdots, m_i - 1$) if $F_S$ is a multiple of $A_i$. Hence $\int_{\partial \Delta} F_S \alpha = 0$ by Cauchy's integral formula. Thus $dF_S$ is in the left kernel $\mathcal{G}_0$ if and only if the function $F_S$ is meromorphic and can be normalized so as to be a multiple of $-D$.

A differential $\alpha \in (P)\Gamma_{\ast e}[ - A ]$ is in the right kernel if and only if
\( \int_{\partial S} S\alpha = 0 \) for all \( S \) which are multiples of \(-B\). Convenient choices for \( S \) show that \( \alpha \) must be a multiple of \( B \), i.e., \( \alpha \in \langle P \rangle \Gamma_{\text{ase}}[D] \).

Since \( \partial \) is finite dimensional we have

\[
\partial / \partial_0 \cong \langle P \rangle \Gamma_{\text{ase}}[-A] / \langle P \rangle \Gamma_{\text{ase}}[D],
\]

or

(6) \hspace{1cm} \dim \partial_0 = \dim \partial - \dim \langle P \rangle \Gamma_{\text{ase}}[-A] / \langle P \rangle \Gamma_{\text{ase}}[D].

Let \( \langle P \rangle M \) be the complex vector space of meromorphic functions on \( W \) whose real and imaginary parts have \( \langle P \rangle L_1 \) behavior near the ideal boundary. Denote by \( \langle P \rangle M[E] \), where \( E \) is a divisor, the subspace of \( \langle P \rangle M \) consisting of functions which are multiples of \( E \). The homomorphism \( d: \langle P \rangle M[-D] \to \partial_0: F_S \to dF_S \) is surjective. The kernel is \( \mathbf{C} \) if \( \text{deg } A = 0 \), and it is zero if \( \text{deg } A \neq 0 \). From (6) we obtain

(7) \hspace{1cm} \dim \langle P \rangle M[-D] = \text{deg } B + 1 - \min(1, \text{deg } A) - \dim \langle P \rangle \Gamma_{\text{ase}}[-A] / \langle P \rangle \Gamma_{\text{ase}}[D].

**Theorem 4.** Let \( A \) and \( B \) be disjoint integral divisors on a Riemann surface \( W \) and let \( D = B - A \). Then (7) is valid for any regular partition \( P \) of the ideal boundary of \( W \).

**Bibliography**


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