\textbf{\textit{W*-ALGEBRAS WITH A SINGLE GENERATOR}}

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In [4] the author set forth a complete set of unitary invariants for a certain class of operators on Hilbert space. The operators considered were exactly those operators which generate a finite \(W^*\)-algebra of type I in the terminology of [2]. One immediately wants to know some nontrivial examples of such operators, and Brown provided several examples in [1]. (Nontrivial here means non-normal operator on infinite-dimensional Hilbert space.) It is the purpose of this note to show that there exists an abundance of such operators, in the sense of the following theorem.

\textbf{Theorem.} If \(R\) is any \(W^*\)-algebra of operators acting on a separable Hilbert space, and \(R\) is of type I, then there exists an operator \(A \in R\) which generates \(R\) (in the sense that \(R\) is the smallest \(W^*\)-algebra containing \(A\)).

We first prove the following lemma.

\textbf{Lemma.} If \(n\) is any cardinal number satisfying \(1 \leq n \leq \aleph_0\), and \(\mathcal{H}\) is any \(n\)-dimensional Hilbert space, then there is an operator \(A\) on \(\mathcal{H}\) such that the \(W^*\)-algebra generated by \(A\) is \(\mathcal{L}(\mathcal{H})\), the algebra of all bounded operators on \(\mathcal{H}\).

\textbf{Proof.} Whether \(n\) is finite or infinite, it clearly suffices to exhibit an operator \(A\) which has no nontrivial reducing subspace. In case \(n\) is finite, take \(A\) to be any operator with \(n\) distinct eigenvalues and with the property that no two eigenvectors corresponding to different eigenvalues are orthogonal. In case \(n = \aleph_0\), choose an orthonormal basis \(\{x_i\}, i = 1, 2, \ldots\), for \(\mathcal{H}\) and define \(A\) by setting \(Ax_i = x_{i+1}\), \(i = 1, 2, \ldots\). That \(A\) has no nontrivial reducing subspace is proved on page 356 of [5].

We now prove the theorem, using von Neumann’s result in [3] that any abelian \(W^*\)-algebra on a separable Hilbert space has a single Hermitian generator and results of Dixmier in [2].

\textbf{Proof of the Theorem.} One knows (see [1] for example) that \(R\) is a direct sum \(\sum_{n \in N} \oplus R_n\) where each \(R_n\) is an \(n\)-homogeneous algebra and \(N\) is some set of cardinal numbers bounded above by \(\aleph_0\). We suppose first that the theorem is known for homogeneous algebras, and return to the proof of this case later. For each \(n \in N\), let \(B_n\) generate \(R_n\), and arrange it so that the \(B_n\) are uniformly bounded in norm.

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Then $B = \sum_{n \in \mathbb{N}} \oplus B_n \subseteq R$. Let $C$ be a generator for the center of $R$. Then one sees immediately that the $W^*$-algebra generated by the pair $(B, C)$ contains each homogeneous algebra $R_n$, and therefore must be $R$. We now obtain a single operator generating $R$ as follows. Write $B = H + iK$, $H$ and $K$ Hermitian. Let $A_1 = A_1^*$ generate the same abelian $W^*$-algebra as the pair $(H, C)$ and let $A_2 = A_2^*$ generate the same algebra as $(K, C)$. Then take $A = A_1 + iA_2$.

We return now to deal with the homogeneous case. Let $R$ be an $n$-homogeneous $W^*$-algebra ($n \leq \mathbb{N}_0$), and let $I$, the unit of $R$, be the identity operator on the separable Hilbert space $\mathcal{H}$. Then $I$ can be written as $I = \sum_{i=1}^n E_i$, where the $E_i$ are mutually orthogonal, equivalent, abelian projections in $R$. Let $\mathcal{K}_1 = E_1(\mathcal{H})$, let $\mathcal{K}_2$ be a Hilbert space of dimension $n$, and let $\mathcal{K} = \mathcal{K}_1 \otimes \mathcal{K}_2$ (the tensor product of $\mathcal{K}_1$ with $\mathcal{K}_2$). It follows from Proposition 5, page 27 of [2], that $R$ is unitarily isomorphic to the (tensor product) $W^*$-algebra $R_1 = E_1RE_1 \otimes \mathcal{L}(\mathcal{K}_2)$ of operators acting on the Hilbert space $\mathcal{K}$, and thus it suffices to obtain a single generator for $R_1$. From von Neumann’s result in [3] we obtain a single generator $C$ for the abelian algebra $E_1RE_1$, and from the lemma we obtain a single generator $B$ for $\mathcal{L}(\mathcal{K}_2)$. Let $G = C \otimes I_{\mathcal{K}_2}$, and let $D = I_{\mathcal{K}_1} \otimes B$. It follows from Proposition 6, page 28 of [2], that the pair $(G, D)$ generates $R_1$, and the argument is completed as above.

Remarks. (1) It is immediate from Exercise 3, page 119 of [2], that one cannot hope to extend this result to algebras of type I on nonseparable spaces.

(2) Is it the case that every $W^*$-algebra (regardless of type) acting on a separable space has a single generator?

**Bibliography**


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