$W^*$-ALGEBRAS WITH A SINGLE GENERATOR

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In [4] the author set forth a complete set of unitary invariants for a certain class of operators on Hilbert space. The operators considered were exactly those operators which generate a finite $W^*$-algebra of type I in the terminology of [2]. One immediately wants to know some nontrivial examples of such operators, and Brown provided several examples in [1]. (Nontrivial here means non-normal operator on infinite-dimensional Hilbert space.) It is the purpose of this note to show that there exists an abundance of such operators, in the sense of the following theorem.

**Theorem.** If $R$ is any $W^*$-algebra of operators acting on a separable Hilbert space, and $R$ is of type I, then there exists an operator $A \in R$ which generates $R$ (in the sense that $R$ is the smallest $W^*$-algebra containing $A$).

We first prove the following lemma.

**Lemma.** If $n$ is any cardinal number satisfying $1 \leq n \leq \aleph_0$, and $\mathcal{H}$ is any $n$-dimensional Hilbert space, then there is an operator $A$ on $\mathcal{H}$ such that the $W^*$-algebra generated by $A$ is $\mathfrak{B}(\mathcal{H})$, the algebra of all bounded operators on $\mathcal{H}$.

**Proof.** Whether $n$ is finite or infinite, it clearly suffices to exhibit an operator $A$ which has no nontrivial reducing subspace. In case $n$ is finite, take $A$ to be any operator with $n$ distinct eigenvalues and with the property that no two eigenvectors corresponding to different eigenvalues are orthogonal. In case $n = \aleph_0$, choose an orthonormal basis $\{x_i\}$, $i = 1, 2, \cdots$, for $\mathcal{H}$ and define $A$ by setting $Ax_i = x_{i+1}$, $i = 1, 2, \cdots$. That $A$ has no nontrivial reducing subspace is proved on page 356 of [5].

We now prove the theorem, using von Neumann's result in [3] that any abelian $W^*$-algebra on a separable Hilbert space has a single Hermitian generator and results of Dixmier in [2].

**Proof of the Theorem.** One knows (see [1] for example) that $R$ is a direct sum $\sum_{n \in \mathbb{N}} \mathfrak{H}_n$ where each $\mathfrak{H}_n$ is an $n$-homogeneous algebra and $\mathbb{N}$ is some set of cardinal numbers bounded above by $\aleph_0$. We suppose first that the theorem is known for homogeneous algebras, and return to the proof of this case later. For each $n \in \mathbb{N}$, let $B_n$ generate $\mathfrak{H}_n$, and arrange it so that the $B_n$ are uniformly bounded in norm.

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Then $B = \sum_{n \in \mathbb{N}} B_n \subset R$. Let $C$ be a generator for the center of $R$. Then one sees immediately that the $W^*$-algebra generated by the pair $(B, C)$ contains each homogeneous algebra $R_n$, and therefore must be $R$. We now obtain a single operator generating $R$ as follows. Write $B = H + iK$, $H$ and $K$ Hermitian. Let $A_1 = A_1^*$ generate the same abelian $W^*$-algebra as the pair $(H, C)$ and let $A_2 = A_2^*$ generate the same algebra as $(K, C)$. Then take $A = A_1 + iA_2$.

We return now to deal with the homogeneous case. Let $R$ be an $n$-homogeneous $W^*$-algebra ($n \leq \aleph_0$), and let $I$, the unit of $R$, be the identity operator on the separable Hilbert space $\mathcal{K}$. Then $I$ can be written as $I = \sum_{i=1}^n E_i$, where the $E_i$ are mutually orthogonal, equivalent, abelian projections in $R$. Let $\mathcal{K}_1 = E_1(\mathcal{K})$, let $\mathcal{K}_2$ be a Hilbert space of dimension $n$, and let $\mathcal{K} = \mathcal{K}_1 \otimes \mathcal{K}_2$ (the tensor product of $\mathcal{K}_1$ with $\mathcal{K}_2$). It follows from Proposition 5, page 27 of [2], that $R$ is unitarily isomorphic to the (tensor product) $W^*$-algebra $R_1 = E_1RE_1 \otimes \mathcal{L}(\mathcal{K}_2)$ of operators acting on the Hilbert space $\mathcal{K}$, and thus it suffices to obtain a single generator for $R_1$. From von Neumann’s result in [3] we obtain a single generator $C$ for the abelian algebra $E_1RE_1$, and from the lemma we obtain a single generator $B$ for $E_1 \mathcal{K}_1 \otimes \mathcal{L}(\mathcal{K}_2)$. Let $G = C \otimes I_{\mathcal{K}_2}$, and let $D = I_{\mathcal{K}_1} \otimes B$. It follows from Proposition 6, page 28 of [2], that the pair $(G, D)$ generates $R_1$, and the argument is completed as above.

Remarks. (1) It is immediate from Exercise 3, page 119 of [2], that one cannot hope to extend this result to algebras of type I on nonseparable spaces.

(2) Is it the case that every $W^*$-algebra (regardless of type) acting on a separable space has a single generator?

Bibliography