ON PRODUCT AND BUNDLED NEIGHBORHOODS

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If a nice space $X$ is embedded in a euclidean space, it may fail to have product neighborhoods; i.e., neighborhoods which are products of $X$ with a ball. However, if the euclidean space is a hyperplane of a higher-dimensional euclidean space one can sometimes guarantee the existence of product neighborhoods in the big euclidean space. For example, it follows from a lemma due to Klee [1] that if $X$ is a $k$-ball in $R^n \subset R^{n+k}$, then $X$ has a neighborhood homeomorphic with $X \times B^n$ in $R^{n+k}$ (where $B^n$ denotes an $n$-ball). We obtain two results along these lines. Theorem 1 gives circumstances in which we get product neighborhoods and Theorem 2 yields ball-bundle neighborhoods. Theorem 2 has been used for smoothing combinatorial manifolds [3].

Definition. We say that a space $X$ has a local multiplication into a space $Z$ if there exists a neighborhood $N$ of the diagonal $\Delta$ of $X \times X$ and a map $\phi: N \to Z$ such that:

(i) $\phi(\Delta) = z_0 \in Z$,
(ii) $\phi|N_x$ is one-to-one, and $\phi(N_x)$ contains a fixed neighborhood $W$ of $z_0$, for all $x$ in $X$. ($N_x = \text{pairs in } N \text{ with first coordinate } x$.)

Example 1. Any topological group $G$ has a local multiplication into itself. Take $N = G \times G$ and define $\phi(g, h) = gh^{-1}$.

Example 2. If $X$ is a $k$-dimensional parallelizable manifold, then $X$ has a local multiplication into $R^k$. We take a Riemannian metric for $X$ and choose $N$ so that if $(x, y) \in N$, then there is a unique geodesic from $x$ to $y$. Let $\tau(x, y)$ denote the vector tangent to this geodesic at $x$ and having length equal to the length of the geodesic. Let $c$ be a cross section of the $k$-frame bundle over $X$. ($c$ exists since $X$ is parallelizable.) We define $\phi(x, y)$ to be the point in $R^k$ with coordinates equal to the dot products of $\tau(x, y)$ with the vectors of $c(x)$. It is easy to verify that $\phi$ is a local multiplication.

A space $S$ is said to have the neighborhood extension property if for any closed subset $B$ of a separable metric space $Y$ and any map $f: B \to S$, there exists an extension of $f$ to some neighborhood of $B$.

Theorem 1. Let $X$ be a compact space which has the neighborhood extension property and has a local multiplication $\phi$ into $Z$. Let $\alpha: X \to H$...
be an embedding of $X$ into a locally compact separable metric group $H$. Then for any sufficiently small compact neighborhood $U$ of the identity $e$ of $H$, there exists a product neighborhood $X \times U$ of $\alpha(X)$ in $H \times Z$.

**Proof.** We will consider that $\alpha$ embeds $X$ into $H \times z_0$ and show that $\alpha$ can be extended to a homeomorphism $\psi$ of $X \times U$ into $H \times Z$. Here $z_0 = \phi(\Delta)$.

Extend $\alpha^{-1}: \alpha(X) \to X$ to a map $\beta$ of a neighborhood $V$ of $\alpha(X)$ into $X$ ($V$ is a neighborhood in $H$). Choose a compact neighborhood $U$ of $e$ in $H$ such that $U \cdot \alpha(X) \subset V$. Let $\hat{x} = \alpha(x)$ and define

$$\psi(x, h) = (h\hat{x}, \phi(\beta(h\hat{x}), x)),$$

One easily checks the following properties of $\psi$.

1. $\psi(x, e) = (\hat{x}, z_0) = \alpha(x)$,
2. $\psi$ is one-to-one and onto a neighborhood.
3. $\psi$ is continuous.

Since $X \times U$ is compact, $\psi$ is a homeomorphism and the theorem is proved.

**Corollary 1.** If $G$ is a group with the neighborhood extension property and $G$ is embedded in $H$, then $G$ has small product neighborhoods $G \times U$ in $H \times G$.

**Corollary 2.** If $X$ is a parallelizable closed $k$-manifold in $\mathbb{R}^n$, then $X$ has a product neighborhood $X \times B^n$ in $\mathbb{R}^{n+k}$.

For example, this is the case if $X$ is a closed orientable 3-manifold or is a compact Lie group. Either Corollary 1 or 2 shows that a simple closed curve $S^1$ in $\mathbb{R}^n$ has a product neighborhood $S^1 \times B^n$ in $\mathbb{R}^{n+1}$. For $k > 1$, the fact that $S^k \subset \mathbb{R}^n$ has a product neighborhood in $\mathbb{R}^{n+k}$ follows from Kleene's lemma and Stallings' unknotting theorem [4]. In general, we would like to show that a $k$-manifold $X$ in $\mathbb{R}^n$ has an $n$-ball-bundle neighborhood in $\mathbb{R}^{n+k}$. We cannot do this, but can get ball-bundle neighborhoods if we are willing to raise the dimension of the embedding space. The next theorem shows how this is done.

Let $X$ be a compact space. We assume the following about $X$ which will be automatically true if $X$ is a smooth manifold: (1) the diagonal $\Delta$ of $X \times X$ has a $k$-ball bundle neighborhood $U$ in $X \times X$. Precisely, we assume that there exists a $k$-plane bundle $\pi = \tau^k$ over $X$ ($= \Delta$) and a homeomorphism $g$ of $U$ onto the vectors of length less than or equal to one (for some metric in $\tau$) such that $g(x, y) \in \tau_x$, $(x, y) \in U$, $\tau_x$ the fibre over $x \in X$. We identify $U$ with $\tau_1$. For $X$ a

\footnote{This paragraph is not fully translated.}
smooth manifold the normal bundle of $\Delta$ (which is the same as the tangent bundle of $X$) gives such a neighborhood.

Since $X$ is compact there exists a vector bundle $v^m$ over $X$ such that $\tau^k \oplus v^m$ is a trivial bundle. (See (2.19) and (2.20) of [2].) Since $\tau^k \oplus v^m$ is trivial, we have a fibre-preserving homeomorphism $\phi: \tau^k \oplus v^m \to X \times \mathbb{R}^{k+m}$, and for $x \in X$ we denote its restriction to the fibre $\tau_x \oplus v_x$ by $\phi_x$.

**Theorem 2.** Suppose $X$ is compact, satisfies (1), and has the neighborhood extension property. If $\alpha: X \to \mathbb{R}^n$ is any embedding of $X$ in $\mathbb{R}^n$, then $\alpha(X)$ has a ball-bundle neighborhood in $\mathbb{R}^n \times \mathbb{R}^{k+m}$.

**Proof.** Let $0^n$ be the product $n$-plane bundle over $X$, and consider that $\alpha$ embeds the zero cross section of $0^n \oplus v^m$. We will prove that there exists $\epsilon > 0$ and an embedding

$$\psi: 0^n \oplus v^m \to \mathbb{R}^n \times \mathbb{R}^{k+m}$$

which extends $\alpha$.

Let $\beta: N \to X$ be an extension of $\alpha^{-1}$ to a neighborhood $N$ of $\alpha(X)$. If $(x, h) \in X \times \mathbb{R}^n$ and $\|h\| < \epsilon$, we can consider $(x, h) \in 0^n$. Let $\bar{x}$ denote $\alpha(X)$ and choose $\epsilon$ small enough so that, for $\|h\| < \epsilon$, $\bar{x} + h \in N$ and $(\beta(\bar{x} + h), x) \in U$.

For $x, y \in X$ we define a map $f_{xy}: \nu_x \to \nu_y$ to be the composition

$$\nu_x \xrightarrow{\text{incl}} \tau_x \oplus \nu_x \xrightarrow{\phi_x} \mathbb{R}^{k+m} \xrightarrow{\phi_y^{-1}} \tau_y \oplus \nu_y \xrightarrow{\text{proj}} \nu_y.$$  

We note that $f_{xy}$ is linear and if $(y, x)$ is in a sufficiently small neighborhood of $\Delta$ in $X \times X$, then $f_{xy}$ is an isomorphism. We assume $\epsilon$ is small enough so that $(\beta(\bar{x} + h), x)$ is in such a neighborhood for $\|h\| < \epsilon$.

For notational convenience let $\sigma = \bar{x} + h$ and $\rho = \beta(\bar{x} + h)$. For $v \in \nu_x$, we define

$$\psi((x, h) + v) = (\sigma, \phi((\rho, x) + f_{xy}(v))).$$

Then $\psi$ maps $0^n \oplus v^m$ into $\mathbb{R}^n \times \mathbb{R}^{k+m}$, and $\psi$ is clearly continuous. It remains to check that $\psi$ is one-to-one.

Suppose $\psi((x_1, h_1) + v_1) = \psi((x_2, h_2) + v_2)$. Then $\sigma_1 = \sigma_2$ so $\rho_1 = \rho_2$ and hence both $(\rho_1, x_1)$ and $(\rho_2, x_2)$ are in the fibre $\tau_{\rho_1}$. Similarly, both $f_{\rho_1x_1}(v_1)$ and $f_{\rho_2x_2}(v_2)$ are in the fiber $\nu_{\rho_1}$. Now $\phi$ is an isomorphism and these fibers are disjoint, so we must have $(\rho_1, x_1) = (\rho_2, x_2)$ so that $x_1 = x_2$ and $h_1 = h_2$. Also $f_{\rho_1x_1}$ is one-to-one so that $v_1 = v_2$ and the theorem is proved.

**Remark 1.** Theorem 2 yields Corollary 2 to Theorem 1 as a special
case, because if $X$ is parallelizable then $\tau$ is trivial and $\nu$ is not needed. Hence $\psi$ embeds $X \times B^k$ into $R^n \times R^k$. More generally

**Corollary 1.** If the diagonal $\Delta$ of $X \times X$ has a $k$-ball bundle neighborhood $\tau$, such that $\tau$ is stably trivial (i.e., $\tau$ plus a trivial bundle is trivial), then $\psi$ embeds $X \times B^{n+1}$ in $R^n \times R^{k+1}$.

**Remark 2.** The proof of Theorem 2 applies to an embedding $\alpha$ of $X$ into any smooth manifold $V^n$, if we replace $\sigma^n$ by the tangent bundle $\tau'$ of $M$ restricted to $X$. Then $\psi$ becomes an embedding of $(\tau'|X), \oplus V^m \to V^n \times R^{k+m}$ which extends $\alpha$.

**References**


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