A NOTE ON SUBGROUPS OF THE MODULAR GROUP

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1. We will follow the notation of [4]. Let $\Gamma$ denote the 2X2 modular group, that is, the set of all 2X2 matrices with rational integral entries and determinant 1. For each positive integer $m$ define $\Gamma(m)$, the principal congruence subgroup of level $m$, by

$$\Gamma(m) = \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid a \equiv d \equiv 1, b \equiv c \equiv 0 \pmod{m} \right\}.$$

Let

$$T_m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

and let $\Delta(m)$ be the normal subgroup of $\Gamma$ generated by $T_m$. That is, $\Delta(m)$ is the smallest normal subgroup of $\Gamma$ containing $T_m$. Clearly, $\Delta(m) \subseteq \Gamma(m)$.

In [4] Reiner considers the following questions raised in [1]:

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(a) Does $\Delta(m) = \Gamma(m)$ for all $m$?

(b) For each $m$, does there exist a positive integer $k$ such that $\Delta(m) \supset \Gamma(mk)$?

He answers both questions in the negative by proving that if $m > 1$ and $m$ is not a prime power, then $\Delta(m)$ does not contain any principal congruence subgroup. The situation when $m$ is a prime power is left open.

The purposes of this note are to point out that the following result is to be found (at least implicitly) in [2] and to give a new proof.

**Theorem.** If $m \geq 6$, then $\Delta(m)$ is of infinite index in $\Gamma(m)$. Since a principal congruence is always of finite index in $\Gamma$, it is a consequence that for $m \geq 6$, $\Delta(m)$ contains no principal congruence subgroup.

I would like to thank Dr. J. R. Smart for calling my attention to [4].

We will make use of the following simple

**Lemma.** $\Delta(m)$ is generated by the set

$$H = \{ X^{-1}T_mX \mid X \in \Gamma \}.$$  

The proof is obtained by noting that the group $G$ generated by $H$ is normal in $\Gamma$ and that $G \subset \Delta(m)$.

Using this lemma we obtain from [2, pp. 267, 354–356] that $\Delta(m) = \Gamma(m)$, for $1 \leq m \leq 5$, and from [2, pp. 356–360] that $\Delta(m)$ is of infinite index in $\Gamma(m)$, for $m \geq 6$.

2. We now give an independent proof of this latter fact based upon the results of [3]. In [3] it was shown that given $m \geq 2$ there exists a function, say $\lambda_1(m; \tau)$, defined and analytic in $\sigma(\tau) > 0$ such that

(i) $\lambda_1(m; X\tau) = \lambda_1(m; \tau) + c(X)$, for each $X \in \Gamma(m)$ and for $\sigma(\tau) > 0$, where $c(X)$ is independent of $\tau$;

(ii) $c(X) = 0$, when $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is parabolic, that is, when $|a + d| = 2$;

(iii) $\lambda_1(m; \tau)$ has a pole of order 1 in the local uniformizing variable at one of the parabolic cusps of $\mathcal{F}_m$, the fundamental region of $\Gamma(m)$, and is regular at all of the other parabolic cusps of $\mathcal{F}_m$.

Of course, (i) and (iii) show that $\lambda_1(m; \tau)$ is an abelian integral connected with $\Gamma(m)$. 

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Now \( T_m \) is parabolic and a simple computation shows that each element of \( H \) is parabolic. But \( c(X_1X_2) = c(X_1) + c(X_2) \) and by (ii) \( c(X) = 0 \), for \( X \in \Delta(m) \). Thus for \( X \in \Gamma(m) \), \( c(X) \) depends only upon the coset of \( X \) modulo \( \Delta(m) \). Thus if \( \Delta(m) \) has finite index in \( \Gamma(m) \), there are only finitely many distinct values \( c(X) \), with \( X \in \Gamma(m) \). But this implies that \( c(X) = 0 \) for all \( X \in \Gamma(m) \). For if there exists \( X_0 \in \Gamma(m) \), with \( c(X_0) \neq 0 \), then \( c(X_0^t) = tc(X_0) \); \( t = 1, 2, 3, \ldots \) provides us with infinitely many distinct values \( c(X) \).

Thus if \( \Delta(m) \) has finite index in \( \Gamma(m) \), \( \lambda_1(m; \tau) \) is an invariant with respect to \( \Gamma(m) \) with precisely one pole of order 1 in \( \mathcal{S}_m \). Since \( \mathcal{S}_m \) is compact, the Riemann-Roch Theorem implies that the genus of \( \mathcal{S}_m \) is zero. By [2, p. 398] \( \mathcal{S}_m \) has genus zero exactly when \( 1 \leq m \leq 5 \). Thus for \( m \geq 6 \), \( \Delta(m) \) has infinite index in \( \Gamma(m) \).

The proof shows that when \( m \geq 6 \), \( \lambda_1(m; \tau) \) cannot be invariant with respect to any subgroup of finite index in \( \Gamma(m) \).

References