A NOTE ON SUBGROUPS OF THE MODULAR GROUP

1963]

MARVIN ISADORE KNOPP

1. We will follow the notation of [4]. Let \( \Gamma \) denote the 2\( \times \)2 modular group, that is, the set of all 2\( \times \)2 matrices with rational integral entries and determinant 1. For each positive integer \( m \) define \( \Gamma(m) \), the principal congruence subgroup of level \( m \), by

\[
\Gamma(m) = \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid a \equiv d \equiv 1, \ b \equiv c \equiv 0 \pmod{m} \right\}.
\]

Let

\[
T_m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}
\]

and let \( \Delta(m) \) be the normal subgroup of \( \Gamma \) generated by \( T_m \). That is, \( \Delta(m) \) is the smallest normal subgroup of \( \Gamma \) containing \( T_m \). Clearly, \( \Delta(m) \subseteq \Gamma(m) \).

In [4] Reiner considers the following questions raised in [1]:

Received by the editors October 6, 1961.

1 Research supported in part by National Science Foundation Grant No. G–14362 at the University of Wisconsin, Madison.
(a) Does $\Delta(m) = \Gamma(m)$ for all $m$?

(b) For each $m$, does there exist a positive integer $k$ such that $\Delta(m) \supset \Gamma(mk)$?

He answers both questions in the negative by proving that if $m > 1$ and $m$ is not a prime power, then $\Delta(m)$ does not contain any principal congruence subgroup. The situation when $m$ is a prime power is left open.

The purposes of this note are to point out that the following result is to be found (at least implicitly) in [2] and to give a new proof.

**Theorem.** If $m \geq 6$, then $\Delta(m)$ is of infinite index in $\Gamma(m)$. Since a principal congruence is always of finite index in $\Gamma$, it is a consequence that for $m \geq 6$, $\Delta(m)$ contains no principal congruence subgroup.

I would like to thank Dr. J. R. Smart for calling my attention to [4].

We will make use of the following simple

**Lemma.** $\Delta(m)$ is generated by the set

$$H = \{X^{-1}T_mX \mid X \in \Gamma\}.$$  

The proof is obtained by noting that the group $G$ generated by $H$ is normal in $\Gamma$ and that $G \subseteq \Delta(m)$.

Using this lemma we obtain from [2, pp. 267, 354–356] that $\Delta(m) = \Gamma(m)$, for $1 \leq m \leq 5$, and from [2, pp. 356–360] that $\Delta(m)$ is of infinite index in $\Gamma(m)$, for $m \geq 6$.

2. We now give an independent proof of this latter fact based upon the results of [3]. In [3] it was shown that given $m \geq 2$ there exists a function, say $\lambda(m; \tau)$, defined and analytic in $s(\tau) > 0$ such that

(i) $\lambda(m; X\tau) = \lambda(m; \tau) + c(X)$, for each $X \in \Gamma(m)$ and for $s(\tau) > 0$, where $c(X)$ is independent of $\tau$;

(ii) $c(X) = 0$, when

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is parabolic, that is, when $|a+d| = 2$;

(iii) $\lambda(m; \tau)$ has a pole of order 1 in the local uniformizing variable at one of the parabolic cusps of $\mathcal{F}_m$, the fundamental region of $\Gamma(m)$, and is regular at all of the other parabolic cusps of $\mathcal{F}_m$.

Of course, (i) and (iii) show that $\lambda(m; \tau)$ is an abelian integral connected with $\Gamma(m)$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Now $T_m$ is parabolic and a simple computation shows that each element of $H$ is parabolic. But $c(X_1X_2) = c(X_1) + c(X_2)$ and by (ii) $c(X) = 0$, for $X \in \Delta(m)$. Thus for $X \in \Gamma(m)$, $c(X)$ depends only upon the coset of $X$ modulo $\Delta(m)$. Thus if $\Delta(m)$ has finite index in $\Gamma(m)$, there are only finitely many distinct values $c(X)$, with $X \in \Gamma(m)$. But this implies that $c(X) = 0$ for all $X \in \Gamma(m)$. For if there exists $X_0 \in \Gamma(m)$, with $c(X_0) \neq 0$, then $c(X_0^t) = tc(X_0)$; $t = 1, 2, 3, \ldots$ provides us with infinitely many distinct values $c(X)$.

Thus if $\Delta(m)$ has finite index in $\Gamma(m)$, $\lambda_1(m; \tau)$ is an invariant with respect to $\Gamma(m)$ with precisely one pole of order 1 in $\mathcal{F}_m$. Since $\mathcal{F}_m$ is compact, the Riemann-Roch Theorem implies that the genus of $\mathcal{F}_m$ is zero. By [2, p. 398] $\mathcal{F}_m$ has genus zero exactly when $1 \leq m \leq 5$. Thus for $m \geq 6$, $\Delta(m)$ has infinite index in $\Gamma(m)$.

The proof shows that when $m \geq 6$, $\lambda_1(m; \tau)$ cannot be invariant with respect to any subgroup of finite index in $\Gamma(m)$.

REFERENCES


THE UNIVERSITY OF WISCONSIN