ON THE CURVATURE OF THE LEVEL LINES OF A HARMONIC FUNCTION

R. P. JERRARD AND L. A. RUBEL

Let $u(z)$ be a harmonic function in some simply connected region W in the complex plane, with grad $u \neq 0$ in W. Let $\Gamma(z_0) = \{z: u(z) = u(z_0)\}$ be the level curve of u that passes through z_0, and let $K(z_0)$ denote the curvature of $\Gamma(z_0)$ at the point z_0.

We study some of the simple properties of $K(z)$. Briefly, log $|K(z)|$ is superharmonic so that $|K(z)|$ satisfies the minimum property. An example shows that $|K(z)|$ need not satisfy the maximum property.

First, we obtain a useful formula for $K(z)$ by selecting a function $w(z) = u(z) + iv(z)$, holomorphic in W, whose real part is u. Then

$$K = |w'| \operatorname{Re}\{w''/(w')^2\}. \tag{1}$$

A related formula (see, for example, Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1, p. 105) has been used extensively in function theory.

To prove (1), we take K as $d\theta/ds$, where s is arc length along $\Gamma(z_0)$, and θ is the angle of inclination of the vector normal to $\Gamma(z_0)$. Hence

$$\theta = -\arg w'[z(s)], \tag{2}$$

$$\frac{d\theta}{ds} = \frac{d}{ds}[\operatorname{Im}(\log w')] = \operatorname{Im}\left[\frac{w''}{w'}\frac{dz}{ds}\right], \tag{3}$$

where primes denote differentiation with respect to z. But along a level curve of u,

$$\frac{dz}{ds} = \frac{dz}{dw}\frac{dw}{ds} = \frac{i}{w'}\frac{dv}{ds} = \pm i\frac{|w'|}{w'}, \tag{4}$$

and (1) follows from (3) and (4) except for the choice of sign, which is a matter of definition.

THEOREM. Log $|K(z)|$ *is superharmonic where* $K(z) \neq 0$.

PROOF. Since $\log|w'(z)|$ is harmonic, it follows from (1) that $\Delta \log |K(z)| = \Delta \log |U(z)|$, where $U = \operatorname{Re}(w''/(w')^2)$ is harmonic. An easy calculation then shows that

$$\Delta \log |K(z)| = -|\operatorname{grad} U|^2/U^2 \leq 0.$$

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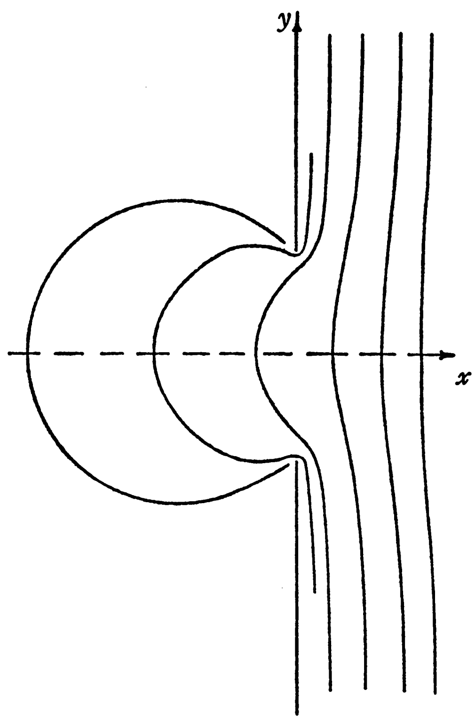
This formula is invalid only when $|w'|=0$ or when $U=0$, and both of these cases have been excluded by hypothesis. q.e.d.

COROLLARY. $|K(z)|$ *has the minimum property. More precisely,*

$$\inf_{z\in W} |K(z)| = \inf_{t\in\partial W} \liminf_{z\to t} |K(z)|. \tag{5}$$

PROOF. Since $\log|K(z)|$ is superharmonic, $|K(z)|$ may not have a local minimum at z_0 unless $K(z_0)=0$. But if $K(z_0)=0$, then $U(z_0)=0$, since it has been required, to begin with, that $|w'|=|\text{grad } u|\neq 0$ in W. Thus, the locus $K(z)=0$ is itself the level line, $U(z)=0$ of the harmonic function U, and by the maximum principle for harmonic functions, this locus extends to the boundary of W.



EXAMPLE. A sketch of the level lines of

$$u(z) = \text{Re}\{z + (z^2+1)^{1/2}\} \tag{6}$$

makes it plausible that $|K(z)|$ can, on the other hand, have a strict maximum at an interior point of W, where in this case W is the z-plane with all points of the y-axis except those for which $|y|<1$ removed. A hydrodynamical interpretation gives these lines as the stream lines of an ideal flow from North to South in the right half plane after the portion of the y-axis from $y=-1$ to $y=+1$ is removed. The sketch indicates that the stream lines are very flat and nearly vertical near the x-axis when $|x|$ is large. This suggests the curvature has a strict local maximum somewhere on the real axis.

For an analytic proof that this is, in fact, the case, we write

$$w = z + (z^2+1)^{1/2} = u + iv, \tag{7}$$

$$z = \frac{1}{2}\left(w - \frac{1}{w}\right), \tag{8}$$

$$(z^2+1)^{1/2} = \frac{1}{2}\left(w + \frac{1}{w}\right), \tag{9}$$

$$w' = w(z^2+1)^{-1/2}, \tag{10}$$

$$w'' = (z^2+1)^{-3/2}. \tag{11}$$

From (1) and (7)–(11), an expression for K in terms of u and v is readily derived.

$$\begin{aligned} K &= |w'| \operatorname{Re} \frac{w''}{(w')^2} = \frac{4}{|w^2+1|^3}\operatorname{Re}(w^3+w) \\ &= \frac{4u(u^2-3v^2+1)}{[(u^2-v^2+1)^2+4u^2v^2]^{3/2}}. \end{aligned} \tag{12}$$

We now show that the anticipated maximum occurs at $p_0=(u_0, v_0) = (1/\sqrt{3}, 0)$, which, in the z-plane, is the point $z_0=-1/\sqrt{3}$. For,

$$\frac{\partial}{\partial v}\log K = -6v\left\{\frac{(u^2+v^2-1)(u^2-3v^2+1)+(u^2-v^2+1)^2+4u^2v^2}{(u^2-3v^2+1)((u^2-v^2+1)^2+4u^2v^2)}\right\}. \tag{13}$$

Since the expression in brackets is positive at (u_0, v_0), it follows that

$$\operatorname{sgn}\frac{\partial K}{\partial v} = -\operatorname{sgn}(v) \tag{14}$$

throughout some small neighborhood W^* of p_0, which is the image of some neighborhood W' of z_0. Hence, for $(u, v)\in W^*$, we have $K(u, v)<K(u, 0)$ unless $v=0$. Furthermore, $K(u, 0)=u(u^2+1)^{-2}$, which has a strict maximum at $u=u_0$. Thus, for $(u, v)\in W^*$, $K(u, v) \le K(u, v_0) \le K(u_0, v_0)$ with equality only if $(u, v)=(u_0, v_0)$, and the assertion is proved.

We conclude with some remarks. If $f(z)$ is a holomorphic (and nonvanishing) function inside the unit disc, then it is easy to construct a holomorphic function $w(z)=u(z)+iv(z)$ for which $f(z)=-1/w'(z)$. Then from (1) we have $K(z)=\operatorname{Re}\{f'(z)/|f(z)|\}$. It then follows that

$$\inf_{|z|<1} \operatorname{Re}\{f'(z)/|f(z)|\} \leq 1, \tag{15}$$

since otherwise the level lines of u would all have curvature exceeding 1 in the unit disc. The following argument shows that this is impossible. Suppose $K>1$ for all level curves of u in $|z|<1$. Then we can find a closed disc D: $|z|\leq 1-\epsilon$ in which $K>1/(1-\epsilon)$. By the strong maximum principle the level curves where u attains its maximum and minimum, say $u=M$ and $u=m$, must be tangent to the circle C: $|z| =1-\epsilon$. At least one of them must be an inner tangent, for if not the sign of K must change as one moves from one such tangent point to another along an arc consisting of subarcs of the level curves $u=$constant and their orthogonal trajectories. But if the curve $u=M$ is an inner tangent to C at P, then near P there are points in the interior of D where $u=M$, and this violates the strong maximum principle. The extremal function $f(z)=-(z-1)^2$ shows that (15) is best possible. There are other proofs of (15).

Another curvature formula for harmonic functions can be obtained from (1). We have

$$K = |w'| \operatorname{Re}\left[\frac{w''}{w'}\frac{dz}{dw}\right] = |w'| \operatorname{Re}\frac{d}{dw}\log w'. \tag{16}$$

If we take the derivative in the direction normal to the level lines of the harmonic function u, then $dv=0$, $du=dw$, and

$$K = |w'| \frac{d}{du}\operatorname{Re}\log w' = |w'|\frac{d}{du}\log |w'| = \frac{d|w'|}{du}. \tag{17}$$

Further, if the normal directional derivative is denoted by d/dn, then $du/dn=|w'|=|\operatorname{grad} u|$, and we have the formula

$$K = \frac{d}{dn}\log |\operatorname{grad} u|. \tag{18}$$

It is easy to integrate K along the orthogonal trajectories to the level curves of u. For example, if such an orthogonal trajectory forms a closed curve, the mean of K along that curve must be zero.

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