A GENERALISATION OF BELLMAN'S LEMMA

B. VISWANATHAM

1. Certain integral inequalities occur very frequently in the theory of ordinary differential equations in various contexts. The aim of this note is to prove a general theorem which is a generalisation of Bellman's Lemma [1], and which also includes other generalisations of the same. We shall also indicate its applications to some problems in ordinary differential equations. We shall first prove the following

THEOREM 1. *If $\phi(x) \leqq \eta + \int_{x_0}^{x} f(s, \phi(s))ds$ where $f(x, y)$ is continuous and monotonic increasing in y in the region R defined by $|x-x_0| \leqq a$; $|y-\eta| \leqq b$, where a and b are positive real numbers; and $\phi(x)$ is continuous in the interval $|x-x_0| \leqq a$, then $\phi(x) \leqq \chi(x)$ where $\chi(x)$ is the maximal solution of the differential equation $z' = f(x, z)$ through (x_0, η) for $x \geqq x_0$. (We shall call this differential equation the associated differential equation of the above integral inequality.)*

PROOF. Take $\phi(x)$ as the zero approximation of the solution of the differential equation $z' = f(x, z)$ through (x_0, η) and set up the successive approximations recursively by

$$\phi_{k+1}^{(x)} = \eta + \int_{x_0}^{x} f(s, \phi_k(s))ds.$$

These successive approximations exist at least on the interval $|x-x_0| \leqq \alpha$ where $\alpha = \min (a, b/M)$ where M is a positive number such that $|f(x, y)| \leqq M$. Further, this sequence of successive approximations is equicontinuous in this interval, for,

$$\begin{aligned} |\phi_n(x_1) - \phi_n(x_2)| &= \left| \int_{x_1}^{x_2} f(s, \phi_{n-1}(s))ds \right| \leqq \int_{x_1}^{x_2} |f(s, \phi_{n-1}(s))|\, ds \\ &\leqq |x_1 - x_2|\, M \leqq \epsilon \qquad \text{if } |x_1 - x_2| \leqq \epsilon/M = \delta. \end{aligned}$$

It is further uniformly bounded because

$$|\phi_n(x)| \leqq |\eta| + M|x_2 - x_1| \leqq \eta + M\alpha.$$

We can show by induction that these successive approximations form a monotonic increasing sequence, for, suppose that $\phi_k^{(x)} \geqq \phi_{k-1}^{(x)}$. Then

Received by the editors August 21, 1961.

$$\phi_{k+1}^{(x)} - \phi_k^{(x)} = \int_{x_0}^{x} \{f(s, \phi_k^{(s)}) - f(s, \phi_{k-1}^{(s)})\} ds \geqq 0$$

since $f(x, z)$ is monotonic increasing in z. Therefore $\phi_{k+1}^{(s)} \geqq \phi_k^{(x)}$. But the basic hypothesis of our theorem is that the zero approximation $\leqq$ first approximation. So the successive approximations form a monotonic increasing, equi continuous, and uniformly bounded function sequence in the interval $|x-x| \leqq \alpha$, and therefore must converge uniformly to a function $\psi(x)$. Further, it is clear that $\psi(x)$ is a solution of the associated differential equation through (x_0, η) and

$$\phi(x) \leqq \psi(x) \qquad \text{for } x_0 \leqq x \leqq x + \alpha.$$

Therefore

$$\phi(x) \leqq \chi(x) \qquad \text{for } x_0 \leqq x \leqq x + \alpha$$

when $\chi(x)$ is the maximal solution through (x_0, η).

As a counterpart to this theorem we can similarly prove the following

THEOREM 2. *Under the same conditions as in Theorem* 1 *if*

$$\phi(x) \geqq \eta + \int_{x_0}^{x} f(s, \phi(s)) ds$$

then $\phi(x) \geqq$ *minimal solution of the associated differential equation through* (x_0, η) *for* $x_0 \leqq x \leqq x + \alpha$.

The proof is the same except that in this case the successive approximations form a monotonic decreasing sequence converging to a solution of the associated equation.

The following may be obtained as corollaries to the above theorems.

COROLLARY 1. *Under the condition of Theorem* 1 *if*

$$\phi(x) \leqq \psi(x) + \int_{x_0}^{x} f(s, \phi(s)) ds$$

then $\phi(x) \leqq \psi(x) + \chi(x)$ *for* $x \geqq x_0$ *where* $\chi(x)$ *is the maximal solution of* $z' = f(x, z + \psi(x))$ *through* $(x_0, 0)$ *as far as this maximal solution exists.*

PROOF. Put $r(x) = \phi(x) - \psi(x)$ and the inequality becomes

$$r(x) \leqq \int_{x_0}^{x} f(s, r^{(s)} + \psi(s)) ds.$$

Apply Theorem 1, and we obtain $r(x) \leqq \chi(x)$. Therefore $\phi(x) \leqq \psi(x) + \chi(x)$. The counterpart to this may be stated as

COROLLARY 2. *Under the conditions of the above theorem if*

$$\phi(x) \geqq \psi(x) + \int_{x_0}^{x} f(s, \phi(s))ds$$

then $\phi(x) \geqq \psi(x) + \chi(x)$ *for* $x \geqq x_0$ *when* $\chi(x)$ *is the minimal solution of the associated equation in Corollary* 1.

Similar theorems may also be proved for intervals with x_0 as the right end point.

2. A very special case of Theorem 1 is what is known as Bellman's Lemma [1] which is as follows:

$$| y(x) | \leqq M + \int_0^x | f(s) | \cdot | y(s) | ds$$

then

$$| y(x) | \leqq M \exp \int_0^x | f(t) | dt.$$

This is obtained by putting $f(x, y) = |f(x)| y$, $x_0 = 0$ and $\eta = M$ in Theorem 1.

Another special case of the same theorem is obtained by putting $f(x, y) = v(x) \cdot g(y)$ where $v(x)$ is non-negative and $g(y)$ is monotonic increasing in y. This case is considered in [2] and also in [3]. It is not necessary to work through the details to prove results of [2] and [3] from Theorem 1.

3. It is clear that in many situations in ordinary differential equations where we use a Lipschitz condition or Lipschitz-like condition we may obtain more general results by applying the above theorems. For example [4] contains the following proposition on approximate solutions.

Suppose f satisfies a Lipschitz condition with Lipschitz constant k; ϕ_1 and ϕ_2 are ϵ_1 and ϵ_2 approximate solutions of the differential equation $x' = f(t, x)$ and for some τ we have $|\phi_1(\tau) - \phi_2(\tau)| \leqq \delta$. Then for $t \geqq \tau$ we have

$$| \phi_1(t) - \phi_2(t) | \leqq \delta e^{k(t-\tau)} + \frac{\epsilon}{k} (e^{k(t-\tau)} - 1) \qquad \text{where } \epsilon = \epsilon_1 + \epsilon_2.$$

If f satisfies the more general condition

$$|f(x, y_1) - f(x, y_2)| \leqq \omega(x, |y_1 - y_2|)$$

where $\omega(x, z)$ satisfies conditions of Theorem 1 we may show easily, applying Theorem 1, that $|\phi_1(t)-\phi_2(t)| \leqq \chi(t)$ for $t \geqq \tau$ where $\chi(t)$ is the maximal solution of $z' = \omega(t, z) + \epsilon$ through (τ, δ). Further the first few examples of page 37 of [4] can all be solved by the application of Corollary 1. Similarly Theorems 1 and 2 can be used in a natural way to extend the results of [3] concerning bounds on the norm of a solution of a differential equation.

References

1. R. Bellman, *Stability theory of differential equations*, McGraw-Hill, London, 1953.

2. I. Bihari, *A generalisation of a lemma of Bellman and its application to uniqueness problems of differential equations*, Acta Math. Acad. Sci. Hungar. **7** (1956), 81–94.

3. C. E. Langenhop, *Bounds on the norm of a solution of a general differential equation*, Proc. Amer. Math. Soc. 11 (1960), 795–799.

4. E. A. Coddington and Norman Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955, pp. 8–9.

Osmania University, Hyderabad, India

THE *G*-FUNCTIONS AS UNSYMMETRICAL FOURIER KERNELS. II

ROOP NARAIN

1. A function $K(x)$ by means of which an arbitrary function $f(x)$ subject to appropriate conditions, is capable of being represented as a repeated integral of the form

$$f(x) = \int_0^\infty K(xu) \int_0^\infty K(uy) f(y) dy du \tag{1.1}$$

has been called a *Fourier kernel* by Hardy and Titchmarsh [1, p. 116]. This is a symmetrical formula. There are also unsymmetrical formulae of the type

$$f(x) = \int_0^\infty K(xu) \int_0^\infty H(uy) f(y) dy du \tag{1.2}$$

in which the kernels in the two integrals are different functions. If $f(t)$ is not continuous at $t = x$, $f(x)$ on the left-hand side of (1.1) or (1.2) is replaced by

$$\frac{1}{2}\{f(x+0) + f(x-0)\}.$$

Received by the editors January 6, 1962.