A GENERALISATION OF BELLMAN’S LEMMA

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1. Certain integral inequalities occur very frequently in the theory of ordinary differential equations in various contexts. The aim of this note is to prove a general theorem which is a generalisation of Bellman’s Lemma [1], and which also includes other generalisations of the same. We shall also indicate its applications to some problems in ordinary differential equations. We shall first prove the following

**Theorem 1.** If \( \phi(x) \leq \eta + \int_{x_0}^{x} f(s, \phi(s))ds \) where \( f(x, y) \) is continuous and monotonic increasing in \( y \) in the region \( R \) defined by \( |x-x_0| \leq a \); \( |y-\eta| \leq b \), where \( a \) and \( b \) are positive real numbers; and \( \phi(x) \) is continuous in the interval \( |x-x_0| \leq a \), then \( \phi(x) \leq \chi(x) \) where \( \chi(x) \) is the maximal solution of the differential equation \( z' = f(x, z) \) through \( (x_0, \eta) \) for \( x \geq x_0 \). (We shall call this differential equation the associated differential equation of the above integral inequality.)

**Proof.** Take \( \phi(x) \) as the zero approximation of the solution of the differential equation \( z' = f(x, z) \) through \( (x_0, \eta) \) and set up the successive approximations recursively by

\[
\phi_{k+1}^{(x)} = \eta + \int_{x_0}^{x} f(s, \phi_k(s))ds.
\]

These successive approximations exist at least on the interval \( |x-x_0| \leq \alpha \) where \( \alpha = \min (a, b/M) \) where \( M \) is a positive number such that \( |f(x, y)| \leq M \). Further, this sequence of successive approximations is equicontinuous in this interval, for,

\[
|\phi_n(x_1) - \phi_n(x_2)| = \left| \int_{x_1}^{x_2} f(s, \phi_{n-1}(s))ds \right| \leq \int_{x_1}^{x_2} |f(s, \phi_{n-1}(s))| ds \\
\leq |x_1 - x_2| M \leq \varepsilon \text{ if } |x_1 - x_2| \leq \varepsilon/M = \delta.
\]

It is further uniformly bounded because

\[
|\phi_n(x)| \leq |\eta| + M |x_2 - x_1| \leq \eta + M\alpha.
\]

We can show by induction that these successive approximations form a monotonic increasing sequence, for, suppose that \( \phi_k^{(x)} \geq \phi_{k-1}^{(x)} \). Then

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\[ \phi^{(k+1)}(x) - \phi^{(k)}(x) = \int_{x_0}^{x} \{ f(s, \phi^{(k)}(s)) - f(s, \phi^{(k-1)}(s)) \} \, ds \geq 0 \]

since \( f(x, z) \) is monotonic increasing in \( z \). Therefore \( \phi^{(k+1)}(x) \geq \phi^{(k)}(x) \). But the basic hypothesis of our theorem is that the zero approximation \( \phi^{(0)}(x) \) is the first approximation. So the successive approximations form a monotonic increasing, equi continuous, and uniformly bounded function sequence in the interval \( |x - x_0| \leq \alpha \), and therefore must converge uniformly to a function \( \psi(x) \). Further, it is clear that \( \psi(x) \) is a solution of the associated differential equation through \( (x_0, \eta) \) and

\[ \phi(x) \leq \psi(x) \quad \text{for} \quad x_0 \leq x \leq x + \alpha. \]

Therefore

\[ \phi(x) \leq \chi(x) \quad \text{for} \quad x_0 \leq x \leq x + \alpha \]

when \( \chi(x) \) is the maximal solution through \( (x_0, \eta) \).

As a counterpart to this theorem we can similarly prove the following

**Theorem 2.** Under the same conditions as in Theorem 1 if

\[ \phi(x) \geq \eta + \int_{x_0}^{x} f(s, \phi(s)) \, ds \]

then \( \phi(x) \geq \) minimal solution of the associated differential equation through \( (x_0, \eta) \) for \( x_0 \leq x \leq x + \alpha \).

The proof is the same except that in this case the successive approximations form a monotonic decreasing sequence converging to a solution of the associated equation.

The following may be obtained as corollaries to the above theorems.

**Corollary 1.** Under the condition of Theorem 1 if

\[ \phi(x) \leq \psi(x) + \int_{x_0}^{x} f(s, \phi(s)) \, ds \]

then \( \phi(x) \leq \psi(x) + \chi(x) \) for \( x \geq x_0 \) where \( \chi(x) \) is the maximal solution of \( z' = f(x, z + \psi(x)) \) through \( (x_0, 0) \) as far as this maximal solution exists.

**Proof.** Put \( r(x) = \phi(x) - \psi(x) \) and the inequality becomes

\[ r(x) \leq \int_{x_0}^{x} f(s, \phi(s)) \, ds. \]

Apply Theorem 1, and we obtain \( r(x) \leq \chi(x) \). Therefore \( \phi(x) \leq \psi(x) + \chi(x) \). The counterpart to this may be stated as

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Corollary 2. Under the conditions of the above theorem if
\[ \phi(x) \geq \psi(x) + \int_{x_0}^{x} f(s, \phi(s)) \, ds \]
then \( \phi(x) \geq \psi(x) + \chi(x) \) for \( x \geq x_0 \) when \( \chi(x) \) is the minimal solution of the associated equation in Corollary 1.

Similar theorems may also be proved for intervals with \( x_0 \) as the right end point.

2. A very special case of Theorem 1 is what is known as Bellman's Lemma [1] which is as follows:
\[
|y(x)| \leq M + \int_{0}^{x} |f(s)| \cdot |y(s)| \, ds
\]
then
\[
|y(x)| \leq M \exp \int_{0}^{x} |f(t)| \, dt.
\]
This is obtained by putting \( f(x, y) = |f(x)|y, x_0 = 0 \) and \( \eta = M \) in Theorem 1.

Another special case of the same theorem is obtained by putting \( f(x, y) = v(x) \cdot g(y) \) where \( v(x) \) is non-negative and \( g(y) \) is monotonic increasing in \( y \). This case is considered in [2] and also in [3]. It is not necessary to work through the details to prove results of [2] and [3] from Theorem 1.

3. It is clear that in many situations in ordinary differential equations where we use a Lipschitz condition or Lipschitz-like condition we may obtain more general results by applying the above theorems. For example [4] contains the following proposition on approximate solutions.

Suppose \( f \) satisfies a Lipschitz condition with Lipschitz constant \( k \); \( \phi_1 \) and \( \phi_2 \) are \( \epsilon_1 \) and \( \epsilon_2 \) approximate solutions of the differential equation \( x' = f(t, x) \) and for some \( \tau \) we have \( |\phi_1(\tau) - \phi_2(\tau)| \leq \delta \). Then for \( t \geq \tau \) we have
\[
|\phi_1(t) - \phi_2(t)| \leq \delta e^{k(t-\tau)} + \frac{\epsilon}{k} (e^{k(t-\tau)} - 1) \quad \text{where} \quad \epsilon = \epsilon_1 + \epsilon_2.
\]
If \( f \) satisfies the more general condition
\[
|f(x, y_1) - f(x, y_2)| \leq \omega(x, |y_1 - y_2|)
\]
where \( \omega(x, z) \) satisfies conditions of Theorem 1 we may show easily.
applying Theorem 1, that $|\phi_1(t) - \phi_2(t)| \leq x(t)$ for $t \geq \tau$ where $x(t)$ is the maximal solution of $z' = \omega(t, z) + \varepsilon$ through $(\tau, \delta)$. Further the first few examples of page 37 of [4] can all be solved by the application of Corollary 1. Similarly Theorems 1 and 2 can be used in a natural way to extend the results of [3] concerning bounds on the norm of a solution of a differential equation.

REFERENCES


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THE G-FUNCTIONS AS UNSYMMETRICAL FOURIER KERNELS. II

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1. A function $K(x)$ by means of which an arbitrary function $f(x)$ subject to appropriate conditions, is capable of being represented as a repeated integral of the form

\begin{equation}
(f) = \int_0^\infty K(xu) \int_0^\infty K(uy)f(y)dydu
\end{equation}

has been called a Fourier kernel by Hardy and Titchmarsh [1, p. 116]. This is a symmetrical formula. There are also unsymmetrical formulae of the type

\begin{equation}
(f) = \int_0^\infty K(xu) \int_0^\infty H(uy)f(y)dydu
\end{equation}

in which the kernels in the two integrals are different functions. If $f(t)$ is not continuous at $t = x$, $f(x)$ on the left-hand side of (1.1) or (1.2) is replaced by

$$\frac{1}{2} \{f(x + 0) + f(x - 0)\}.$$