FUNCTION ALGEBRAS WITH CLOSED RESTRICTIONS

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Let $C(X)$ be the (supremum normed) algebra of all complex continuous functions on a compact Hausdorff space $X$, and $A$ a closed subalgebra containing the constants and separating the points of $X$. For $F \subseteq X$ let $f|F$ be the restriction of the function $f$ to $F$, $A|F = \{ f|F : f \in A \}$. The purpose of this note is to prove the following result, which characterizes $C(X)$ among its closed separating subalgebras.

**Theorem.** Suppose that for every closed $F \subseteq X$, $A|F$ is closed in $C(F)$. Then $A = C(X)$.

Our proof leans heavily on some facts obtained recently by Katznelson [1]. In fact little else, beyond the Riesz representation theorem, is needed (for some facts concerning Banach algebras the reader is referred to [2; 3]).

Let $F$ be a closed subset of $X$. $kF = \{ f : f \in A, f(F) = 0 \}$, the kernel of $F$, is an ideal in $A$, and we can form the quotient algebra $A/kF$; for an element $f + kF (f \in A)$ of $A/kF$, $\| f + kF \|$ will denote the usual quotient norm (inf $\{ \| f + g \| : g \in kF \}$) under which $A/kF$ is a Banach algebra. For $x \in F$, $f + kF \rightarrow f(x)$ is a multiplicative linear functional on $A/kF$, so necessarily of norm $\leq 1$, whence $\| f|F \| = \sup f(F) \leq \| f + kF \|$, where $\| f|F \|$ is of course the norm inherited by the closed subalgebra $A|F$ of $C(F)$. Now clearly $f + kF \rightarrow f|F$ is an isomorphism of $A/kF$ onto $A|F$, and so, by the open mapping theorem, topological: there is a constant $c_F$ for which

$$\| f + kF \| \leq c_F \| f|F \|, \quad f \in A,$$

a relation which we shall need somewhat later.

To begin the proof proper we shall show that for closed disjoint subsets $F$ and $K$ of $X$ there is an $f$ in $A$ which is 1 on $F$, 0 on $K$. Indeed let $x$ and $y$ be distinct elements of $X$. Since $A$ separates $X$, there is an $f$ in $A$ with $f(x) = 0, f(y) = 1$, so we can find closed neighborhoods $V_x$ of $x$ and $W_y$ of $y$ for which $|f(V_x)| \leq 0.5, \quad 1 - f(W_y) | \leq 0.5$. As is well known there is a sequence $\{ p_n \}$ of polynomials in one complex variable which converge uniformly on $\{ z : |z| \leq 1 \} \cup \{ z : |1 - z| \leq 0.5 \}$ to the characteristic function of $\{ z : |z| \leq 0.5 \}$; consequently

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\{(p_n \circ f) | (V_x \cup W_x) \}\ is a sequence of elements of \(A\) which converges uniformly on \(V_x \cup W_x\) to 1 on \(V_x\), 0 on \(W_x\). Since \(A\) is closed under uniform convergence by hypothesis, there is an \(e \in A\) which is 1 on \(V_x\), 0 on \(W_x\).

Now let \(y\) be a fixed element of \(K\). \(F\) can be covered by finitely many \(V_x\), \(x \in F\), say \(V_{x_1}, \ldots, V_{x_n}\). If \(e_1, \ldots, e_n\) are the corresponding elements of \(A\), then (as in a familiar argument) clearly \(e_1' = e_1 + e_2 - e_1 e_2\) is 1 on \(V_{x_1} \cup V_{x_2}\) and 0 on the neighborhood \(W_{x_1} \cap W_{x_2}\) of \(y\); similarly \(e_2' = e_1' + e_3 - e_1' e_3\) is 1 on \(V_{x_1} \cup V_{x_2} \cup V_{x_3}\) and 0 on \(W_{x_1} \cap W_{x_2} \cap W_{x_3}\). Continuing we thus obtain an element \(f_y\) of \(A\) which is 1 on \(F\), and vanishes on some neighborhood \(U_y\) of \(y\).

But now finitely many \(U_y\) cover \(K\), so that the product \(f\) of the corresponding \(f_y\) yields an element of \(A\) which is 1 on \(F\) and 0 on \(K\), as desired.

Alternatively stated, we have found an idempotent in \(A/k(K \cup F)\) which separates \(F\) and \(K\). What we now require is a bound on the norms of such idempotents, and to obtain this we repeat an argument of Katznelson's [1, Lemmas 3, 4, 5]. Let us say \(A\) is bounded on a subset \(V\) of \(X\) if there is a constant \(C_V\) for which, whenever \(F\) is closed and \(F \subseteq V\), any idempotent in \(A/kF\) has norm \(<C_V\). (Our next lemmas are taken from Katznelson [1], and are repeated here for the convenience of the reader.)

**Lemma 1.** Let \(V_1\) and \(V_2\) be open in \(X\), and \(A\) be bounded on each \(V_i\). Then \(A\) is bounded on every closed subset \(F\) of \(V_1 \cup V_2\).

**Proof.** Since \(F \setminus V_2\) is a closed subset of \(V_1\) we can find an open \(W_1 \supset F \setminus V_2\), \(W_1 \subseteq V_1\); and since \(F \setminus W_1 \subseteq V_x\) there is an open \(W_2 \supset F \setminus W_1\), \(W_2 \subseteq V_x\). \(F \setminus W_1\) and \(F \setminus W_2\) are disjoint closed subsets of \(X\) so there is a \(\phi\) in \(A\) for which \(\phi(F \setminus W_1) = 0, \phi(F \setminus W_2) = 1\), as we have seen.

Let \(P\) be a closed subset of \(F\), and let \(f + kP\) be an idempotent in \(A/kP\), so \(f\) is zero or 1 at each point of \(P\). Then \(f + k(P \cap W_i)\) is an idempotent in \(A/k(P \cap W_i)\) and so (by the definition of the quotient norm) we can choose \(f_i \in f + k(P \cap W_i)\) with \(\|f_i\| < C_{V_i}\), \(P \cap W_i\) being a closed subset of \(V_i\), \(i = 1, 2\). But now \(f = \phi f_1 + (1 - \phi) f_2\) on \(P\), since \(P \subseteq F \subseteq W_1 \cup W_2\), and thus

\[
\|f + kP\| \leq \|\phi f_1 + (1 - \phi) f_2\| < \|\phi\|C_{V_1} + \|1 - \phi\|C_{V_2}
\]

so the right side may be taken as \(C_P\).

Still following Katznelson let us say \(A\) is bounded at \(x \in X\) if \(A\) is bounded on \(V\), for some neighborhood \(V\) of \(x\). Then as an obvious consequence of Lemma 1 and compactness we have

**Lemma 2.** If \(F\) is closed and \(A\) is bounded at each \(x\) in \(F\) there is an
open $V \supset F$ on which $A$ is bounded.

**Lemma 3.** There are at most finitely many $x$ in $X$ at which $A$ is not bounded.

**Proof.** If there were infinitely many we could find a sequence $\{x_n\}$ of them with disjoint open neighborhoods $\{V_n\}$ for which $A$ is not bounded on $V_n$. So each $V_n$ would contain a closed $F_n$ for which there is an idempotent $f_n + kF_n$ in $A/kF_n$ of norm $\geq n$.

Now let $F = (\bigcup F_n)^{-}$; since $V_n \cap F_m = \emptyset$ for $n \neq m$, $F = F_n \cup (\bigcup_{m \neq n} F_m)^{-}$, $F_n \cap (\bigcup_{m \neq n} F_m)^{-} = \emptyset$ and thus there is a $\phi$ in $A$ with $\phi(F_n) = 1$, $\phi(F \setminus F_n) = 0$. Consequently $\phi f_n$ is zero or 1 at each $x$ in $F$, and $\phi f_n + kF$ is an idempotent in $A/kF$; but

$$n \leq \|f_n + kF_n\| = \||\phi f_n + kF_n\| \leq \|\phi f_n + kF\|$$

(since $kF \subset kF_n$) so the idempotents in $A/kF$ are not bounded in norm. By (1) they must be bounded, and the lemma is proved.

Let $\{x_1, \ldots, x_n\}$ be the finite set given by Lemma 3 and let $F$ be any closed set in $X$ disjoint from $\{x_1, \ldots, x_n\}$. By Lemma 2, $A$ is bounded on a neighborhood of $F$; so there is a constant $C_F$ for which, whenever $K$ is closed, $K \subset F$, each idempotent in $A/kK$ has norm $< C_F$.

Now suppose $\mu$ is a (complex regular Borel) measure on $F$ orthogonal to the closed subalgebra $A \mid F$ of $C(F)$, and let $K$ be a closed subset of $F$, $\epsilon > 0$. Choose a closed $K_0 \subset F \setminus K$ for which the total variation of $\mu$ on $(F \setminus K) \setminus K_0$ is less than $\epsilon/C_F$, and then choose $f \in A$ which is 1 on $K$, 0 on $K_0$, and of norm $< C_F$. Then

$$0 = \int f d\mu = \int_K 1 d\mu + \int_{K_0} 0 d\mu + \int_{(F \setminus K) \setminus K_0} f d\mu,$$

so $|\mu(K)| = |\int_{(F \setminus K) \setminus K_0} f d\mu| \leq C_F \cdot \epsilon/C_F = \epsilon$. Thus since $\epsilon > 0$ is arbitrary, $\mu K = 0$ for every closed $K \subset F$, and $\mu$ itself vanishes by regularity. Consequently $A \mid F = C(F)$.

Now this implies any measure $\mu$ orthogonal to $A$ is carried by $\{x_1, \ldots, x_{n}\}$. For if not $\mu \neq 0$ for some closed $F$ disjoint from $\{x_1, \ldots, x_{n}\}$ by regularity, and if $V$ is an open neighborhood of $\{x_1, \ldots, x_{n}\}$ with $V \cap F = \emptyset$, we can choose $f \in A$ with $f(V^-) = 0$, $f(F) = 1$; then $f \mu$ (the measure corresponding to the functional $g \rightarrow \int g f d\mu$ on $C(X)$) is a measure carried by $X \setminus V$, and orthogonal to $A$ and so orthogonal to $A \mid (X \setminus V) = C(X \setminus V)$, hence the zero measure. But $f \mu(F) = \int f d\mu = \mu F \neq 0$, the desired contradiction, and $\mu$ orthogonal to $A$ must be carried by $\{x_1, \ldots, x_{n}\}$.
Since $A$ separates the points of $X$ this is clearly impossible unless all orthogonal measures vanish, i.e., unless $A = C(X)$, which completes the proof of the theorem.

References


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