A NOTE ON PARACOMPACTNESS AND NORMALITY OF MAPPING SPACES

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In a discussion with the author, J. C. Moore asked whether the space $Y^X$ of mappings of a compact metric space $X$ in a locally compact paracompact space $Y$ is necessarily paracompact. (Throughout, "space" means "Hausdorff space," all mappings are understood to be continuous, and $Y^X$ is given the compact-open topology.) It has been shown by Fadell [1] that if we drop the assumption that $Y$ is locally compact the answer is negative. We shall give two simple examples to show that not even the assumption that $Y$ is compact is sufficient. In both examples, as in Fadell's, $X$ will be the unit interval $I$, so that $Y^X$ becomes the space of paths on $X$. In our first example, $Y$ is 1-dimensional (in the covering sense); in our second, $Y$ has infinite dimension but $Y^X$ is not even normal. In Fadell's example, $Y$ (though not compact) is 1-dimensional and $Y^X$ is also not normal. It would be interesting to know whether $Y^I$ is normal whenever $Y$ is both compact and finite-dimensional.

Let $L$ be the transfinite line from 0 to $\omega_1$ inclusive; this is the compact, totally ordered 1-dimensional space obtained by joining each countable ordinal to its successor by an interval, and adjoining $\omega_1$ (see, for example, [2, p. 263] for a formal definition of $L = (\omega_1)$).

**Theorem 1.** $L^I$ is not paracompact.

Let $M$ be the subset of $L^I$ consisting of those maps $f: I \rightarrow L$ for which $f(0) = 0$. Clearly $M$ is closed in $L^I$, and it will suffice to prove $M$ not paracompact. In what follows, $\alpha$ will run over the ordinals $< \omega_1$. Put $U_\alpha = \{ f | f \in M \text{ and, for all } t \in I, f(t) < \alpha \}$; the sets $U_\alpha$ form an open covering of $M$. Let $\mathcal{U}$ be any (open) refinement of $\{ U_\alpha \}$; we show that $\mathcal{U}$ is not locally finite.

We define a sequence of maps $f_n \in M$ and of countable ordinals $\alpha(n)$ ($n = 1, 2, \ldots$) as follows. Take $f_1 \in M$ such that $f_1(1) > 1$. Then $f_1 \in$ some $V_1 \in U_1$, and $V_1 \subset$ some $U_{\alpha(1)}$. When $f_n$ and $\alpha(n)$ have been defined, take $f_{n+1} \in M$ such that (i) $f_{n+1}(1) > \alpha(n)$, (ii) $f_{n+1}(t) \geq f_n(t)$ for all $t \in I$, and (iii) $f_{n+1}(t) = f_n(t)$ whenever $0 \leq t \leq 1 - n^{-1}$. This is possible, because the subspace of $L$ consisting of points $\leq \alpha(n) + 1$ is order-homeomorphic to $I$. Then take $V_{n+1} \in \mathcal{U}$ containing $f_{n+1}$, and

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choose $\alpha(n+1)$ so that $V_{n+1} \subseteq U_{\alpha(n+1)}$. Clearly $\alpha(1) < \alpha(2) < \cdots$;
put $\sup \alpha(n) = \beta(= \omega_1)$. It is easy to see that, for each $t \in I$, $\lim_{n \to \infty} f_n(t)$ exists; call it $g(t)$. Thus $g(0) = 0$ and $g(1) = \beta$. Further, $g$ is continuous, because (a) if $0 \leq t < 1$ there is a neighborhood of $t$ on which $f_n$ is ultimately constant, from (iii), and (b) if $t = 1$ and $m$ is a given positive integer, we have $f_{n+1}(s) > \alpha(m)$ whenever $s$ is close enough to 1, and then $\alpha(n) > f_n(s) > \alpha(m)$ for all $n > m$, proving $\beta \geq g(s) \geq \alpha(m)$. By Dini's theorem [3, p. 239] $f_n \to g$ uniformly, and hence $f_n \to g$ in $M$. Thus every neighborhood of $g$ meets $V_n$ for infinitely many values of $n$. But the sets $V_n$ are all distinct, since if $k > n$ we have $V_n \subseteq U_{\alpha(n)}$ and $f_k \not\subseteq V_k - U_{\alpha(n)}$, by (i). Hence $U$ is not locally finite, and $M$ is not paracompact.

I do not know whether $L^1$ is normal. However, a compact (but infinite-dimensional) space $Y$ for which $Y^1$ is not normal can be obtained from the following lemmas:

1. For any space $X$ and family of spaces $Y_n$, $(\prod X_n)^X$ and $\prod(Y_n^X)$ are homeomorphic.

This is well known, and the proof is straightforward.

2. If a product of nonempty spaces is normal, all but at most countably many of the factor spaces are countably compact.

This is [4, Theorem 3, Corollary].

3. $I^1$ is not countably compact.

Again this is known; the proof consists in observing that the collection of maps $f_n$ ($n = 1, 2, \cdots$), where $f_n(t) = t^n$, has no cluster point (for the pointwise limit is discontinuous).

**Theorem 2.** Let $Y$ be the product of uncountably many copies of the unit interval. Then $Y^1$ is not normal.

For, by (1), $Y^1$ is homeomorphic to $\prod(I_n^1)$, which by (2) and (3) is not normal.

The construction employed here raises the question: Under what conditions will $Y^X$ be compact? The following partial answers are easily obtained.

**Theorem 3.** A necessary and sufficient condition that $Y^1$ be compact is that $Y$ is compact and contains no arc.

We may identify $Y$ with a closed subset of $Y^1$, namely the set of constant functions; hence, if $Y^1$ is compact, so is $Y$. If $Y$ contains an arc $J$, then $J^1$ is a closed subset of $Y^1$, and (3) above shows that $Y^1$ cannot be compact. Conversely if $Y$ contains no arc, then each $f \in Y^1$ must be constant; for $f(I)$ is a Peano space (it is metrizable because, $Y$ being Hausdorff and $I$ compact, $f$ is closed) and so contains an
arc if it has more than one point. Thus $Y^I$ and $Y$ are homeomorphic, which proves the sufficiency.

It is easily seen that in Theorem 3 we may replace "compact" by "countably compact" throughout, and $I$ (in either version) by any normal arcwise connected space of more than one point. Also the condition remains necessary if $I$ is replaced by any connected completely regular space of more than one point.

Finally, a similar argument but with $I$ replaced by the space $K$ formed by a single convergent sequence (plus its limit point) proves that the only spaces $Y$ for which $Y^X$ is compact for all compact metric spaces $X$ (or for $X = K$) are the spaces with at most one point.

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References


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