Given a sequence \( \{a_n\}_{n=0}^\infty \) of complex numbers, a number of theorems have been proved concerning implications of the vanishing of certain differences \( \Delta^n a_n \) if the given sequence satisfies some growth restriction. The first such result, proved by Agnew [1], states that if \( \{a_n\} \) is bounded and \( \Delta^2 a_0 = 0 \) for all \( n \), then \( a_n = 0 \) for all \( n \). If all the odd differences are zero, the sequence is constant. Fuchs [6] proved the following: Let \( a_n = o(n^k) \) for some positive number \( k \), and let \( n_j \) be a subsequence of the positive integers such that if \( n(R) \) is the number of \( n < R \), then \( n(R) \geq R/2 \) for \( R > R_0 \). If \( \Delta^n a_0 = 0 \) for all \( n_j \), then \( a_n = p(n) \) where \( p(x) \) is a polynomial of degree less than \( k \).

Buck [4] assumed only that \( \lim \sup |a^n|^{1/n} < 1 \) and \( \Delta^n a_0 = 0 \) for all \( n \) belonging to a set of positive integers of density \( d > \frac{1}{2} \) and proved there is a function \( f \) of exponential type whose growth function \( h(\theta) \) satisfies \( h(\pm \pi/2) < \pi \) such that \( f(n) = a_n \) for all \( n \). In this paper, we show that if the given sequence is extended to \( \{a_n\}_{n=\infty}^{\infty} \) by letting \( a_{-n} = a_n \), then the vanishing of certain of the even central differences \( \Delta^{2n} a_{-n} \) has similar implications. Or, letting \( a_{-n} = -a_{n-1} \), vanishing of odd differences \( \Delta^{2n-1} b_{-n} \) gives similar results.

If \( G \) is a connected set, let \( K[G] \) denote the class of all entire functions of exponential type whose conjugate indicator diagrams \( D(f) \) are contained in \( G \). If \( G \) is the rectangle \( \{x + iy | x| \leq a; |y| \leq c \} \), then \( K[a, c] \) will be used for \( K[G] \). Let \( C_n \) denote the polynomial \( z(z-1) \cdots (z-n+1)/n! \).

Certain results concerning the sequence \( \{\mathcal{L}_n\} \) of Stirling functionals given by \( \mathcal{L}_n(f) = \Delta^nf(-n/2) \) will be needed. These functionals

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have the representation
\[ \Delta^n f(-n/2) = \frac{1}{2\pi i} \int_{\Gamma} (e^{i\xi} - e^{-i\xi})^n F(\xi) \, d\xi \]
where \( F \) is the Borel (Laplace) transform of \( f \) and \( \Gamma \) is any simple contour enclosing the conjugate indicator diagram \( D(f) \). Let \( B \) be the set of all \( \xi \) satisfying \( |e^{i\xi} - e^{-i\xi}| < 2 \). Then \( B \) is a convex, lens-shaped region, symmetric about the origin whose boundary has vertices at \( \pm \pi i \) and crosses the real axis at \( \pm \log(3+2\sqrt{2}) \). For \( f \) in \( K[B] \), Buck \cite{3} showed that
\[ f(z) = \sum_{n=0}^{\infty} \Delta^n f(-n/2)(z/n)C_{e+n/2,n-1,n-1}, \]
convergent for all \( z \). The author \cite{5} showed that for a given sequence \( \{c_n\} \) of complex numbers, there is a function \( f \) in \( K[B] \) such that \( \Delta^n f(-n/2) = c_n \); \( n = 0, 1, 2, \ldots \) if and only if \( \limsup |c_n|^{1/n} < 2 \). If we let \( G(t) = \sum c_n t^n \), then \( f \) has the representation
\[ f(z) = \frac{1}{2\pi i} \int_{B} \frac{G(t)}{t} \exp\left[2z \sinh^{-1} \frac{t}{2}\right] dt \]
where \( E \) is a simple contour contained in the region of regularity of \( G \) and enclosing the disk \( |t| \leq \frac{1}{2} \). Then the conjugate indicator diagram \( D(f) \) is contained in the convex hull of the image of \( E \) under the map \( \xi = 2 \sinh^{-1} 1/(2z) \).

**Theorem 1.** Let \( \{b_n\}_{n=-\infty}^{\infty} \) be an even sequence of complex numbers such that \( \limsup |b_n|^{1/n} \leq 1 \). Suppose there is a set \( A \) of positive integers of density \( d > 0 \) such that for all \( n \) in \( A \), \( \Delta^2 b_{-n} = 0 \). Then
\[ \sum \Delta^2 b_{-n}(z/2n)C_{e+n/2,n-1,n-1} \]
converges to an even function \( f \) in \( K[B] \) and \( f(n) = b_n; n = 0, \pm 1, \pm 2, \ldots \).

We need the following lemma.

**Lemma.** For a sequence \( \{b_n\}_{n=-\infty}^{\infty} \), define \( Q(t) \) formally by

\[ Q(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} C_{n,k} b_{-n-2k} t^n \]

and define \( P(z) \) by \( P(z) = (1+2z)^{-1}Q(z/(1+2z)) \). Then formally \( P(z) = \sum \Delta^2 b_{-n} z^n \).

**Proof.**\(^1\) Let \( E \) be an operator defined for a sequence \( a = \{a_k\} \) by \( E(a)(k) = a(k+1) \). Then \( \Delta = E - 1 \), and we have

\(^1\) This proof was suggested by Professor R. C. Buck.
\begin{align*}
Q(t) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} C_{n,k} b_{-n-2k} t^n \\
&= \sum_{n=0}^{\infty} t^n E^{-n} \sum_{k=0}^{n} C_{n,k} E^{2k} b_0 \\
&= \sum_{n=0}^{\infty} t^n (E + E^{-1})^n b_0 \\
&= \frac{1}{1 - t(E + E^{-1})} b_0.
\end{align*}

\begin{align*}
Q(z/(1 + 2z)) &= (1 + 2z) \frac{1}{1 - z(E - 2 + E^{-1})} b_0 \\
&= (1 + 2z) \sum_{n=0}^{\infty} z^n (E - 2 + E^{-1})^n b_0 \\
&= (1 + 2z) \sum_{n=0}^{\infty} \Delta^n b_{-n} z^n.
\end{align*}

\textbf{Proof of Theorem 1.} Since

\[ \limsup \left| b_n \right|^{1/n} \leq 1, \quad \limsup \left| \sum_{k=0}^{n} C_{n,k} b_{-n+2k} \right| \leq 2. \]

Thus \( Q(t) \) is regular in the disk \( |t| < \frac{1}{2} \). Then from its definition \( P(z) \) is regular for \( |z/(1 + 2z)| < \frac{1}{2} \) or \( |z| < |z + \frac{1}{2}| \) which is the set of all \( z \) whose real part is greater than \( -\frac{1}{2} \). Let \( G(z) = P(z^2) = \sum \Delta^n b_{-n} z^{2n} \). Then \( G \) is regular for all \( z \) such that \( \Re(z^2) > -\frac{1}{4} \); i.e., for \( z \) in the region containing the origin and bounded by the equilateral hyperbola \( y^2 - x^2 = \frac{1}{4} \) where \( z = x + iy \). Let \( r_0 \) be the radius of convergence of \( \sum \Delta^n b_{-n} z^{2n} \). Then \( r_0 \geq 1 \). Since \( G \) is even, \( G(z) = \sum c_n z^n \) where \( c_{2n+1} = 0 \) and \( c_{2n} = \Delta^n b_{-n} \). Then from the hypothesis, \( c_n = 0 \) for all \( n \) belonging to a set of density \( d' > \frac{1}{2} \). Thus, by Pólya's density theorem \([7]\), \( G \) has a singularity on every arc of \( |z| = r_0 \) of opening \( 2\pi(1 - d) \) and this is less than \( \pi \). But if \( r_0 = \frac{1}{2} \), the only possible singularities are at \( i/2 \) or \(-i/2\); so \( r_0 > \frac{1}{2} \). Then, by the results on Stirling functionals quoted earlier, there is a function \( f \) in \( K[B] \) such that \( \Delta^{2n} f(-n) = \Delta^{2n} b_{-n} \) for all \( n \), and \( f(z) = \sum \Delta^{2n} b_{-n} (z/2n) C_{z+n-1,n-1} \) convergent for all \( z \). Since \( (z/2n) C_{z+n-1,n-1} \) is even for each \( n \), \( f \) is even. It can be shown by induction that \( f(n) = b_n; \ n = 0, \pm 1, \pm 2, \cdots \), using the fact that \( \Delta^{2n} f(-n) = \Delta^{2n} b_{-n} \) for each \( n \). Q.E.D.

\textbf{Theorem 2.} In Theorem 1, if \( d \leq \frac{1}{2} \), then \( f \) is of zero type.

\textbf{Proof.} If \( d \geq \frac{1}{2} \), then \( G(z) = \sum \Delta^n b_{-n} z^{2n} \) has zero coefficients for
all \( n \) belonging to a set of positive integers of density at least \( \frac{3}{4} \). Then, by Pólya's density theorem, \( G \) has a singularity on every arc of its circle of convergence of opening \( \pi/2 \). But \( G \) is regular for all \( z = r e^{i\theta} \) with \( \theta \leq \pi/4 \), so \( G \) is entire. Then in representation (1) of \( f \), the contour \( E \) can be taken as a circle of arbitrarily large radius, so that its image under the map \( \xi = 2 \sinh^{-1} 1/(2t) \) can be made to lie in an arbitrarily small disk about the origin. Therefore \( D(f) \) is the origin, i.e., \( f \) is of zero type. Q.E.D.

**Corollary 3.** In Theorem 1, if \( d \geq \frac{1}{2} \) and \( b_n = o(n^k) \) as \( n \to \infty \) for some \( k > 0 \), then \( f \) is a polynomial of degree less than \( k \).

**Proof.** The function \( f \) is of zero type and since \( f \) is even, \( f(n) = o(|n|^k) \) as \( n \to \pm \infty \); so \( f \) is a polynomial of degree less than \( k \) [2, p. 183].

**Corollary 4.** In Corollary 3, if \( \{b_n\} \) is bounded then it is a constant sequence.

Thus we have obtained theorems analogous to those of Buck, Fuchs, and Agnew referred to at the beginning.

Since, for an odd sequence \( \{c_n\}_{n=-\infty}^{\infty} \), \( \Delta^2 c_{-n} = 0 \) for all \( n \), we have the following:

**Corollary 5.** If any bounded sequence \( \{b_n\}_{n=-\infty}^{\infty} \) has \( b_0 = 0 \) and \( \Delta^2 b_{-n} = 0 \) for all \( n \) belonging to a set of positive integers of density \( d \geq \frac{1}{2} \), then \( \{b_n\} \) is an odd sequence.

For a sequence \( \{b_n\}_{n=-\infty}^{\infty} \) such that \( b_{-n} = -b_{n-1}; n = 1, 2, 3, \ldots \), we obtain theorems identical with those above except that the even differences are replaced by odd differences \( \Delta^2 b_{-n} \) and the interpolating function is an odd function. The proofs of these theorems are almost the same as the proofs of the above theorems.

**Bibliography**


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