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INVARIANT SUBSPACES IN L^1

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1. Denote by L^1 and L^2 the spaces of summable and square summable functions on the circle group and by H^1 and H^2 the subspaces of L^1 and L^2 consisting of those functions whose Fourier coefficients vanish for negative indices. A subspace M of L^1 or of L^2 is said to be invariant if

$$\chi \cdot M \subseteq M$$

and to be doubly invariant if also

$$\bar{\chi} \cdot M \subseteq M$$

where χ is the character

$$\chi(e^{ix}) = e^{ix}.$$

H^1 and H^2 are closed subspaces which are invariant but not doubly invariant.

Invariant subspaces on the circle were originally studied by Beurling [1] who showed that the closed invariant subspaces contained in H^2 are of the form $\phi \cdot H^2$ where ϕ is a function in H^2 which has modulus one a.e. Such functions are called inner functions. Rudin and de Leeuw [3, p. 476] have shown that the closed invariant subspaces in H^1 have the same structure as those in H^2. That is to say, they are of the form $\phi \cdot H^1$ where again ϕ is an inner function. The arguments given in [1; 3] depend to a considerable extent on the function theory of the spaces H^1 and H^2.

Recently, Helson and Lowdenslager [2] using Hilbert space methods have given a very elegant and simple proof of Beurling's theorem.

Received by the editors December 22, 1961.

They establish more than Beurling did in that they show that the closed invariant subspaces of L^2 which are not doubly invariant are of the form $\phi \cdot H^2$ where ϕ is a function which has modulus one a.e. We wish to emphasize that the Helson-Lowdenslager argument is free of function-theoretic considerations.[1]

The purpose of this note is to show that a few algebraic and topological considerations together with a description of the closed invariant subspaces of L^2 which are not doubly invariant leads to a description of the corresponding subspaces of L^1. One reason for our interest in this is that from a description of these invariant subspaces one can recover a considerable amount of the function theory of H^1 and H^2.[2]

2. THEOREM 1. *The closed invariant subspaces of L^1 which are not doubly invariant are of the form $\phi \cdot H^1$ where ϕ is a function which has modulus one a.e.*

PROOF. Let M be a closed invariant subspace of L^1 which is not doubly invariant. Suppose for the moment that M contains no doubly invariant subspaces other than the subspace consisting only of the zero vector.

Let $f \in M$ and assume f is not the zero vector. We claim

$$f = \phi g \tag{1}$$

where $g \in H^1$, $\phi \in M$, and ϕ has modulus one a.e. Indeed, let

$$f = f_1 f_2 \tag{2}$$

be any factorization of f as the product of functions in L^2, and consider the closed invariant subspaces M_1 and M_2 in L^2 generated by f_1 and f_2: M_k $(k=1, 2)$ is the closed linear span in L^2 of the functions $\chi^n f_k$ where $n=0, 1, \cdots$. Since M is invariant and closed in L^1,

$$M_1 \cdot M_2 \subseteq M. \tag{3}$$

In particular $f_1 \cdot M_2$ is contained in M, and since M does not contain any nontrivial doubly invariant subspaces, M_2 cannot be doubly invariant. Similarly M_1 is not doubly invariant. Therefore by Beurling's theorem (Helson-Lowdenslager version)

$$M_k = \phi_k \cdot H^2 \qquad (k = 1, 2) \tag{4}$$

[1] The argument given in [2] appeals to the fact that a function in H^1 which vanishes on a set of positive measure must vanish a.e. However it is not difficult to see that the argument goes through without using this.

[2] This was first pointed out to the author by Professor Helson.

where ϕ_k has modulus one a.e. (1) now follows from (2), (3), and (4).

Now denote by J the set of all functions in M which have modulus one a.e. (because of (1) J is not empty), and let N be the closed invariant subspace in L^2 generated by the set J: N is the closed linear span in L^2 of the set of functions $\chi^n\phi$ where $\phi\in J$ and $n\geqq 0$. Since the L^2 norm dominates the L^1 norm, N must be contained in M. This in turn implies that N is not doubly invariant, and therefore

$$N = \phi\cdot H^2 \tag{5}$$

where ϕ has modulus one a.e.

We now claim $M=\phi\cdot H^1$. Since ϕ belongs to N, ϕ also belongs to M, and therefore the invariance of M implies $\phi\cdot H^1\subseteq M$. On the other hand if $f\in M$, then from (1) $f=\psi g$ where $\psi\in J$ and $g\in H^1$. (5) implies $\psi=\phi\lambda$ where $\lambda\in H^2$. Since λ must have modulus one a.e., $\lambda g\in H^1$, and therefore f, which is equal to $\phi\lambda g$, is in $\phi\cdot H^1$.

We have proved Theorem 1 under the assumption that M contains no doubly invariant subspaces other than the subspace consisting only of the zero vector. Before we remove this condition on M we need to observe that a function in M which vanishes on a set of positive measure must vanish a.e. To this end let f be a nonzero function in M and denote by M_f the closed invariant subspace in L^1 generated by f. Then M_f is contained in M, and since M does not contain any nontrivial doubly invariant subspaces, the same must be true of M_f. Therefore $M_f=\phi\cdot H^1$. Since $\phi\in M_f$, there is a sequence g_n of trigonometric polynomials such that g_nf converges to ϕ in the metric of L^1. Since ϕ has modulus one a.e., f cannot vanish on any set of positive measure.

In particular, since H^1 contains no nontrivial doubly invariant subspaces, a function in H^1 which vanishes on a set of positive measure must vanish a.e. It is this well known fact that we need to complete the proof of Theorem 1.

Suppose now that M is invariant and that M contains a nontrivial doubly invariant subspace N.[3] Denote by K the annihilator of M: K consists of all functions $g\in L^\infty$ such that

$$\int fgd\sigma = 0^4$$

for all $f\in M$. If $f\in N$ and $g\in K$, the double invariance of N implies

[3] The argument which follows is a variation of one which appears in [2, p. 253].

[4] $\int fgd\sigma=(1/2\pi)\int_{-\pi}^{\pi}f(e^{ix})g(e^{ix})dx$.

$$\int \chi^n fg d\sigma = 0$$

for all n, and therefore fg vanishes a.e. Since N contains at least one function which does not vanish on some set of positive measure, every $g \in K$ must vanish on a set of positive measure. If now $f \in M$ and $g \in K$, the invariance of M implies

$$\int \chi^n fg d\sigma = 0$$

for $n \geqq 0$, and therefore $fg \in H^1$. Since g vanishes on a set of positive measure, fg must vanish a.e., and hence

$$\int \bar{\chi} fg d\sigma = 0$$

for all $g \in K$. This implies that $\bar{\chi} f \in M$, and M is doubly invariant.

We have shown that a closed invariant subspace of L^1 which is not doubly invariant cannot contain any nontrivial doubly invariant subspaces, which completes the proof of Theorem 1.

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