ON A PROBLEM OF C. BERGE

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The unsolved problem, number 4 on page 251 in [1] states: "Does the sum of two graphs have a kernel (French: noyau) if each of them has a kernel?" The purpose of this note is to give a negative answer. The definitions and notations used here are the same as in [1].

Let \( G_1 = (X_1, \Gamma_1) \) where \( X_1 = \{x_1, x_2, x_3, x_4\} \), and \( \Gamma_1 x_1 = \{x_3, x_4\} \), \( \Gamma_1 x_2 = \{x_3, x_4\} \), \( \Gamma_1 x_3 = \{x_1, x_4\} \), and \( \Gamma_1 x_4 = \emptyset \). Also let \( G_2 = (X_2, \Gamma_2) \) where \( X_2 = \{y_1, y_2\} \) and \( \Gamma_2 y_1 = \{y_2\} \) and \( \Gamma_2 y_2 = \emptyset \). Clearly, \( G_1 \) has a kernel, namely \( \{x_3\} \), and \( G_2 \) has \( \{y_2\} \) as its kernel. Form \( G = G_1 + G_2 = (X_1 \times X_2, \Gamma) \). We claim that \( G \) does not have a kernel. Suppose \( G \) had one, denoted by \( S \), then \( (x_4, y_2) \) must belong to \( S \), because \( \Gamma(x_4, y_2) = \emptyset \). By definition of \( S \), none of the nodes in \( \Gamma^{-1}(x_4, y_2) = \{(x_1, y_2), (x_2, y_2), (x_3, y_2), (x_4, y_2)\} \) can be in \( S \). The rest of nodes of \( X_1 \times X_2 \), \( (x_1, y_1) \), \( (x_2, y_1) \) and \( (x_3, y_1) \), generate a complete subgraph (it is also an odd directed cycle), only one of them can be in \( S \). But, no matter which one of them is in \( S \), there is always another one, \( (x, y) \), of them which has the property \( \Gamma(x, y) \cap S = \emptyset \) where \( (x, y) \in S \). This is a contradiction to the definition of \( S \). Hence, \( G \) does not have a kernel.

Similarly, one can construct a family of such graphs: Take \( G'_1 \) to be a complete directed graph of \( n \) nodes \( (n \geq 3) \) with a Hamiltonian cycle (or take \( G'_1 \) to be a directed cycle of \( n \) nodes where \( n \) is odd and \( > 1 \)), and take \( G_1 \) to be \( G'_1 \cup \{x_{n+1}\} \) such that from every node of \( G'_1 \) there is a directed edge toward the node \( x_{n+1} \) and no edge leads from \( x_{n+1} \). Take \( G_2 \) as before. Then each of \( G_1 \) and \( G_2 \) has a kernel, but \( G = G_1 + G_2 \) does not have one by the similar argument as before.

Reference


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