THE DIRECT PRODUCT OF RIGHT SINGULAR SEMI-
GROUPS AND CERTAIN GROUPOIDS

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1. Introduction. Let $G$ be a semigroup which satisfies the two axioms:

   P 1.1. There is at least one (left identity) $e \in G$ such that $ea = a$ for all $a \in G$.

   P 1.2. For every $a \in G$ and for every left identity $e \in G$, there is at least one $b \in G$ such that $ab = e$.

Such systems were investigated by A. H. Clifford [1] and H. B. Mann [2]. According to their results, $G$ is the direct product of a group and a right singular semigroup, i.e., a semigroup in which $xy = y$ for all $x, y$. Clifford called such systems multiple groups, Mann called them $(l, r)$ systems, but we call them right groups.

A generalization of right groups, namely, the direct product of a right singular semigroup and a semigroup with a two-sided identity, led us to a more general system in which a weakened associative law holds and to which we apply the name $M$-groupoid.

In this paper we prove that an $M$-groupoid is the direct product of a right singular semigroup and a groupoid with a two-sided identity and investigate how defining conditions for $M$-groupoids compare with those for right groups.

2. Orthogonal decompositions of groupoids. Let $A$ and $B$ be groupoids in Bruck's sense [4]. A groupoid $S$ is the direct product of $A$ and $B$, written $A \times B$, if and only if $S$ is the set of all ordered pairs $(a, b)$, $a \in A$, $b \in B$, under the binary composition $(a, b)(c, d) = (ac, bd)$.

A decomposition of a groupoid $S$ is a partition of $S$, $S = \biguplus_{a \in A} S_a$, $S_a \cap S_b = \emptyset$, $a \neq b$, in which for every $\alpha, \beta \in \Gamma$ there is $\gamma \in \Gamma$ such that $S_\alpha S_\beta \subset S_\gamma$. Two decompositions, $S = \biguplus_{a \in A} S_a$, $S = \biguplus_{\lambda \in \Gamma'} T_\lambda$, of a groupoid $S$ are said to be orthogonal if and only if for each $\alpha \in \Gamma$ and for each $\lambda \in \Gamma'$, $S_\alpha \cap T_\lambda = \{x_\alpha\}$, a set containing exactly one element.

Immediately we have the following lemma:

**Lemma 1.** A groupoid $S$ is isomorphic to the direct product of two groupoids if and only if $S$ has two orthogonal decompositions.

Clifford introduced this notion and associated terminology in his paper but did not apply the principle directly. We use a restricted
version of this lemma which K. Shoda [3] established in the theory of direct decompositions using lattice theoretic methods. Our form of the principle is easily proved using elementary methods.

3. The Main Theorem. An $M$-groupoid $S$ is a groupoid which satisfies the following conditions:

P 3.1. There is at least one $e \in S$ such that $ex = x$ for all $x \in S$.

P 3.2. If $y$ or $z$ is a left identity of $S$, then $(xy)z = x(yz)$ for all $x \in S$.

P 3.3. For any $x \in S$, there is a unique left identity $e$ (which may depend on $x$) such that $xe = x$.

Theorem 1. An $M$-groupoid $S$ is the direct product of a right singular semigroup and a groupoid with a two-sided identity, and conversely. The groupoid with the two-sided identity is obtained as $Se$ where $e$ is a left identity.

Proof. We find two orthogonal decompositions of $S$ and apply Lemma 1. Let $e \in S$ be a left identity and consider $Se$. The mapping $\eta: S \rightarrow Se$, $\eta x = xe$, is a surjective homomorphism. For any $se \in Se$,

$$ (se)\eta = (se)e = s(ee) = se. $$

By P 3.1, P 3.2, $(xy)\eta = (xy)e = (x(ey))e = ((xe)y)e = (xe)(ye) = (x\eta)(y\eta)$ for any $x, y \in S$. From the homomorphism $\eta$ we obtain a decomposition of $S$:

$$ (3.5) \ S = \bigcup_{se \in Se} S_{se}, \ S_{se} \cap S_{s' e} = \emptyset, \ s \neq s', \ S_{se} = \{ x \in S: xe = se \}. $$

For $x \in S$ let $e_x \in S$ be the unique left identity such that $xe_x = x$, and consider the mapping $\beta: S \rightarrow R$, $x\beta = e_x$ where $R$ is the subset of left identities. $\beta$ is also a surjective homomorphism. If $xe_x = x$, $ye_y = y$, then, by P 3.2, $(xy)e_x = x(ye_y) = xy$. Thus, by P 3.3, $(xy)\beta = e_x = e_x e_y = (x\beta)(y\beta)$. From the homomorphism we get the decomposition:

$$ (3.6) \ S = \bigcup_{u \in R} T_u, \ T_{u_1} \cap T_{u_2} = \emptyset, \ u_1 \neq u_2; \ T_u = \{ x \in S: xu = x \}. $$

(3.5) and (3.6) are orthogonal. Let $S_t$ and $T_u$ be arbitrary cosets of the decompositions (3.5) and (3.6) respectively. Since $(zu)e = z(ue) = ze = z$, where $z \in S_t$, and $(zu)u = z(uu) = zu$, $zu \in S_t \cap T_u$. By the definitions of $S_t$ and $T_u$, $ye = z$ and $yu = y$ so that $y = yu = (ye)u = (ye)u$. Thus, $S_t \cap T_u$ contains exactly one element $zu$.

By (3.4), $Se$ is a groupoid with a two-sided identity $e$. Also, $R$ is a right singular semigroup. By Lemma 1, $S$ is isomorphic to the direct product of $R$ and $Se$.

Conversely, it follows easily that the direct product $R' \times S'$
4. **Right groups.** A right group $S$ is a groupoid which satisfies:

P 4.1. For every $x, y, z \in S$, $(xy)z = x(yz)$.

P 4.2. For any $a, b \in S$, there is a unique $c \in S$ such that $ac = b$.

A right group is a special case of an $M$-groupoid.

**Theorem 2.** A right group $S$ is isomorphic to the direct product of a right singular semigroup and a group.

**Proof.** It is sufficient to show that P 3.1 and P 3.3 are fulfilled and that $Se, e \in S$ is a left identity, is a group.

By P 4.2, for any $a \in S$, there is a unique $c \in S$ such that $ac = a$. Since $S$ is a groupoid, we have, for any $x \in S$, that $(ac)x = ax$. By P 4.1, $a(cx) = ax$. An application of the uniqueness of P 4.2 to $a(cx) = ax$ yields $cx = x$ for all $x \in S$. Thus, $e \in S$ is a left identity and P 3.1, P 3.2, P 3.3 hold.

Next we shall prove that $Se$ is a group. Clearly $e$ is a right identity in $Se$. By P 4.2, for any $ae \in Se$, there is $c \in S$ such that $(ae)c = e$. But then, for $ce \in Se$, $(ae)(ce) = ((ae)c)e = ee = e$. Thus $ce$ is a right inverse of $ae$ with respect to $e$. Now, Theorem 1 implies Theorem 2.

We now list some conditions which the elements of a groupoid $S$ may satisfy, and from this list we formulate sets of conditions which when imposed on a groupoid $S$ characterize a right group.

P 4.1. For all $a, b, c \in S$, $(ab)c = a(bc)$.

P 4.2. For all $a, b \in S$ there exists exactly one $c \in S$ such that $ac = b$.

P 4.3. $ab = ac$ implies $b = c$.

P 4.4. For all $a, b \in S$ there exists $c \in S$ such that $ac = b$.

P 4.5. There exists $e \in S$ such that for all $a \in S$, $ea = a$.

P 4.6. For all $a \in S$ and for all left identities $e \in S$ there is $c \in S$ such that $ac = e$.

P 4.7. For all $a \in S$ there is a left identity $e \in S$ and $c \in S$ such that $ac = e$.

P 4.8. For all $a \in S$ there is a left identity $e \in S$ and $c \in S$ such that $ca = e$.

**Theorem 3.** If $S$ is a groupoid which satisfies any of the following sets of conditions, then $S$ is a right group and conversely.

I. \{P 4.1, P 4.2\},

II. \{P 4.1, P 4.3, P 4.4\},

III. \{P 4.1, P 4.4, P 4.5\},

IV. \{P 4.1, P 4.5, P 4.6\},

V. \{P 4.1, P 4.5, P 4.7\},
VI. \{ P 4.1, P 4.5, P 4.8 \},
VII. \( S \cong R \times G, \) \( R \) is a right singular semigroup, \( G \) is a group.

Although these formulations are proved equivalent in [1], we want to point out that this can be done in the following way.

I \( \rightarrow \) VII \( \rightarrow \) II \( \rightarrow \) III \( \rightarrow \) IV \( \rightarrow \) V \( \rightarrow \) VI \( \rightarrow \) I.

We also have the following additional conditions:
P 4.9. If \( e \) and \( f \) are idempotents, then \( ef = f \).
P 4.10. \( S \) is the set union of some groups; that is, \( S = \bigcup \alpha G_\alpha \), each \( G_\alpha \) is a group.

VIII. \{ P 4.1, P 4.9, P 4.10 \}.

**Theorem 4.** A groupoid \( S \) satisfies VIII if and only if \( S \) is a right group.

**Proof.** Verification of VIII \( \rightarrow \) VI and VII \( \rightarrow \) VIII is the way we establish this.

VIII \( \rightarrow \) VI. Let \( e_a \) be the identity of \( G_\alpha \) and let \( x \in S \) be an arbitrary element. By P 4.10, there is a \( \beta \) such that \( x \in G_\beta \). Let \( e_\beta \) be the identity of \( G_\beta \). By P 4.9,

\[
(4.11) \quad e_ax = e_a(e_\beta x) = (e_\alpha e_\beta)x = e_\beta x = x.
\]

Thus \( e_a \) is a left identity for \( S \).

Again, by P 4.10, if \( y \in S \), then there is \( \gamma \) such that \( y \in G_\gamma \). Let \( y^{-1} \in G_\gamma \) be the inverse of \( y \) with respect to the identity \( e_\gamma \in G_\gamma \) then \( y^{-1} \) and \( e_\gamma \) satisfy P 4.8 since \( G_\gamma \) is a group.

VII \( \rightarrow \) VIII. Since right singular semigroup \( R \) and group \( G \) are both associative, their direct product is associative. Thus, P 4.1 holds. The only idempotents in \( R \times G \) are pairs of the form \( (y, e) \), where \( y \) is arbitrary and \( e \in G \) is the identity. For such elements, \( (x, e)(y, e) = (xy, ee) = (y, e) \) and P 4.9 holds. Any element of \( R \times G \) has the form \( (y, a) \), \( y \in R, a \in G \). For a fixed \( y \in R \), let \( G_y \) be the subset of \( R \times G \) in which the first element of each ordered pair is \( y \). Clearly, \( G_y \) is isomorphic to \( G \). Thus, \( R \times G = \bigcup_{y \in R} G_y \) and P 4.10 holds.

**Corollary 1.** If a semigroup \( S \) is the set union of some right groups \( S_\alpha \) and \( ef = f \) for all idempotents \( e \) and \( f \), then \( S \) is a right group.

This is an immediate consequence of Theorem 4.

5. **Conditions for \( M \)-groupoids.** In this section we investigate whether or not the characterizations of right groups in §4 when properly modified yield characterizations of \( M \)-groupoids. If it is not already included, adjoin "there is at least one left identity \( e_a \)" to the characterizations I through VI and replace associativity by the
Weakened associative law P 3.2. Denote the new systems by I' through VI', respectively. The following counterexample shows that there are M-groupoids which do not satisfy I' through VI'; that is, I' through VI' are not necessary conditions.

\[
\begin{array}{cccccc}
  a & b & c & d & e & f \\
  a & a & b & c & d & e & f \\
  b & b & c & c & e & f & f \\
  c & c & b & b & f & e & e \\
  d & a & b & c & d & e & f \\
  e & b & c & c & e & f & f \\
  f & c & b & b & f & e & e \\
\end{array}
\]

With respect to sufficiency we have that I' and II' are sufficient conditions, IV' through VI' are not sufficient, but we have not yet answered this question for III'.

**Theorem 5.** I'→{P 3.1, P 3.2, P 3.3}.

**Proof.** It is only necessary to verify P 3.3. By P 4.2, for any \( a \in S \) there is exactly one \( e \in S \) such that \( af = a \). For any left identity \( e \in S \), \( ae = (af)e = a(f)e \). By P 4.2, \( ae = a(f)e \) yields \( fe = e \). Now, for any \( x \in S \), \( fx = f(ex) = (f)e(x) = ex = x \). Thus, \( f \) is a left identity in \( S \).

That II'→I' is clear since P 4.3 and P 4.4 imply P 4.2.

The following example satisfies IV', V', VI' but it is not an M-groupoid.

\[
\begin{array}{cccc}
  a & b & c & d \\
  a & a & b & c & d \\
  b & b & a & a & a \\
  c & b & a & a & a \\
  d & b & a & a & a \\
\end{array}
\]

We now consider the following modifications of VIII.

P 5.1. Existence of left identity.

P 5.2. \((xy)z = x(yz)\) if \( y \) or \( z \) is a left identity.

P 5.3. If \( e, f \) are idempotents, \( ef = f \).

P 5.4. \( S \) is the union of disjoint groupoids each of which has a two-sided identity.

P 5.5. There is a decomposition \( \{ S_\alpha \} \) of \( S \) such that each \( S_\alpha \) is a groupoid with a two-sided identity.

**VIII'**. \{P 5.1, P 5.2, P 5.3, P 5.4\}. 
VIII4'. \{P 5.1, P 5.2, P 5.3, P 5.5\}.

VIII4' does not imply \{P 3.1, P 3.2, P 3.3\}. The following example shows this.

\[
\begin{array}{cccccc}
  & a & b & c & d & e & f \\
 a & a & b & c & d & e & f \\
b & b & c & c & b & c & c \\
c & c & b & b & c & b & b \\
d & a & b & c & d & e & f \\
e & e & f & f & e & f & f \\
f & f & e & e & f & e & e \\
\end{array}
\]

This multiplication table satisfies VIII4', but it is not an \(M\)-groupoid since P 3.3 does not hold; that is, \(a, d\) are left identities for which \(ba = b\) and \(bd = b\).

However we have:

**Theorem 6.** VIII4' characterizes an \(M\)-groupoid.

**Proof.** Necessity is clear since an \(M\)-groupoid is the direct product of a right singular semigroup and a groupoid with a two-sided identity according to Theorem 1.

For the proof of sufficiency, we may show that VIII4' implies P 3.3. By the equalities of (4.11), a two-sided identity \(e_a\) of \(S_a\) is a left identity of \(S\). Conversely, any left identity \(e \in S\) is a left identity of some \(S_a\) and hence it coincides with the two-sided identity of \(S_a\). Now, by P 5.5, for any \(x \in S\), there is \(\alpha\) such that \(x \in S_{\alpha}\), and so \(xe_a = x\). Suppose that \(xe_\beta = x\) for some left identity \(e_\beta \in S\), where \(e_\beta \in S_\beta\), \(\beta \neq \alpha\). Immediately we have \(S_\alpha S_\beta \subset S_\alpha\) by the assumption concerning decompositions; this contradicts P 5.3 since \(e_\alpha e_\beta = e_\beta\). Thus P 3.3 holds.

**References**