SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

A NOTE ON CONTINUED FRACTIONS

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It is well known that the convergents $p_n/q_n$ of a continued fraction of a real number $\alpha$ are its best approximations, i.e. that for every rational $a/b \neq p_n/q_n$ with $1 \leq b \leq q_n$ and $n \geq 1$ there is

$$| q_n \alpha - p_n | < | b \alpha - a | .$$

(1)

The usually produced proofs of this fact use rather intricate arguments. Here is a proof of (1) based on the two following elementary properties of the continued fraction

(i) $1/q_{n+1} < | q_{n-1} \alpha - p_{n-1} | < 1/q_n,$

(ii) $q_n | q_{n-1} \alpha - p_{n-1} | + q_{n-1} | q_n \alpha - p_n | = 1.$

If $a/b = p_{n-1}/q_{n-1}$ inequality (1) holds by (i):

$$| q_n \alpha - p_n | < 1/q_{n+1} < | q_{n-1} \alpha - p_{n-1} | .$$

If $| a q_{n-1} - b p_{n-1} | \geq 1$ then

$$| a/b - \alpha | + | \alpha - p_{n-1}/q_{n-1} | \geq | a/b - p_{n-1}/q_{n-1} | \geq 1/(bq_{n-1})$$

i.e.

$$b | q_{n-1} \alpha - p_{n-1} | + q_{n-1} | b \alpha - a | \geq 1,$$

while the assumption $1 \leq b \leq q_n$ implies by (ii)

$$1 \geq b | q_{n-1} \alpha - p_{n-1} | + q_{n-1} | q_n \alpha - p_n |$$

whence

$$| q_n \alpha - p_n | \leq | b \alpha - a | .$$

(1')

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Equality in (1') is for irrational $\alpha$ impossible in view of $a/b \neq p_n/q_n$ assumed. For rational $\alpha$ excluding equality in (1') requires, strangely enough, additional argument which may run as follows. Substitute into (1') with equality presumed

$$\alpha = \frac{P}{Q} = \frac{p_{n-1}r_n + p_{n-2}}{q_{n-1}r_n + q_{n-2}}$$

with rational $r_n$, and $Q \geq q_n$, to get

$$|bP - aQ| = |r_n - a_n|,$$

where $a_n$ is the $n$th partial quotient in the continued fraction expansion of $\alpha$. The last equality shows that $r_n$ is an integer, thus, by Euclid's algorithm, $r_n = a_n$ whence $a/b = P/Q = p_n/q_n$ contrary to assumption.