ON THE $\omega^*$-SEQUENTIAL CLOSURE OF A CONE

R. D. McWILLIAMS

1. If $X$ is a real Banach space and $X^{**}$ the second conjugate space of $X$, then for each subset $A$ of $X^{**}$ let $K_{\omega^*}(A)$ be the $\omega^*$-sequential closure of $A$ in $X^{**}$; thus $F \in K_{\omega^*}(A)$ if and only if there is a sequence $\{F_n\}$ in $A$ such that $F(f) = \lim_{n} F_n(f)$ for all $f \in X^*$. If $J_X$ is the canonical mapping from $X$ into $X^{**}$, then $K_{\omega^*}(J_X X)$ is closed in the norm topology of $X^{**}$ [1]. In the present paper it will be shown that if $P$ is a norm-closed cone in $X$, then $K_{\omega^*}(J_X P)$ is norm-closed in $X^{**}$, but $K_{\omega^*}(K_{\omega^*}(J_X P))$ need not be norm-closed in $X^{**}$.

2. If $s$ is a bounded real function on $[0, 1]$, let $\|s\| = \sup_{0 \leq t \leq 1} |s(t)|$. If $\mathcal{S} = \{s_k\}$ is a bounded sequence of functions, i.e., if each $s_k$ is bounded and $\sup_{u \in u} \|s_u\|$ is finite, let $\|\mathcal{S}\| = \sup_{u \in u} \|s_u\|$. If $Q$ is a set of functions, let $L(Q)$ be the set of all functions $x$ such that $x$ is the pointwise limit of a bounded sequence in $Q$. Let $\alpha$ be the collection of all double sequences $A = \{a_{ik}\}$ of non-negative numbers such that for each $k$ the following conditions are satisfied: (1) $a_{ik} = 0$ for all $i < k$ and for all but a finite number of $i \geq k$; and (2) $\sum_i a_{ik} = 1$. If $\mathcal{S} = \{s_k\}$ is a sequence of functions and $A \in \alpha$, let $T_A S$ be the sequence $\{s_{ik}\}$ such that $s_{ik} = \sum_i a_{ik} s_{i}$ for each $k$. It is clear that (1) $T_A$ is a linear operator; (2) if $\mathcal{S}$ is bounded, then so is $T_A S$ and $\|T_A S\| \leq \|S\|$; (3) if $\mathcal{S}$ is pointwise convergent, then $T_A S$ is pointwise convergent to the same limit.

Lemma 1. If $R$ and $S$ are bounded sequences in the space $C[0, 1]$ of continuous real functions on the interval $[0, 1]$ and $R$ and $S$ converge pointwise to functions $r$ and $s$ respectively, then for each $\epsilon > 0$ there exists $A \in \alpha$ such that $\|T_A R - T_A S\| < \|r - s\| + \epsilon$.

Proof. The sequence $R - S$ is bounded and converges pointwise to $r - s$. Hence by [1; proof of Lemma 1] there exists for each $\epsilon > 0$ an $A$ having the required property.

Lemma 2. Let $P$ be a cone in $C[0, 1]$, let $\{z_n\}$ be a sequence in $L(P)$, and let $z$ be a bounded function such that $\lim_{n \to \infty} \|z_n - z\| = 0$. Then $z \in L(P)$.

Proof. It may be assumed that $\|z_n - z\| < 2^{-n}$ for each $n$. For each

Presented to the Society, February 22, 1962 under the title $\omega^*$-sequential closure of a cone in a Banach space; received by the editors December 4, 1961.
there is a bounded sequence \( S_n = \{ s_{nk} \} \) in \( P \) which converges point-wise to \( z_n \); by Lemma 1 it may be assumed that \( \| S_n \| < \| z \| + 2^{-n} \).

By induction on \( n \) it can now be shown that for each \( n \) and for each \( i \leq n \) there exists a bounded sequence \( S_{in} = \{ s_{ink} \} \) in \( P \) such that

1. \( \lim_{k \to \infty} s_{ink}(t) = z_i(t) \) for each \( t \in [0, 1] \),

2. \( \lim_{k \to \infty} s_{ink}(t) = z_i(t) \) for each \( t \in [0, 1] \),

and

3. \( \| S_{in} - S_{jn} \| < 2^{-i} \) if \( i < j \leq n \).

For \( n = 1 \), the sequence \( S_{11} \) can be taken to be \( S_1 \). For \( n > 1 \), if \( S_{i, n-1} \) has been obtained for each \( i \leq n - 1 \), Lemma 1 may be applied \((n - 1)\) times in succession to obtain \( A_1, \ldots, A_{n-1} \subseteq \mathcal{A} \) such that

\[
\| T_{A_i} T_{A_{i-1}} \cdots T_{A_1} (S_{i, n-1} - S_n) \| < 2^{-i}
\]

for each \( i < n \). The induction step is completed by letting \( S_{in} = T_{A_i} T_{A_{i-1}} \cdots T_{A_1} S_{i, n-1} \).

Now let \( S = \{ s_k \} \) be the bounded sequence in \( P \) such that \( s_k = s_{kkk} \) for each \( k \). Let \( t \in [0, 1] \) and \( i \in \omega \). Then for every \( k \geq i \),

\[
| s_k(t) - z(t) |
\]

is equal to

\[
| s_{kkk}(t) - s_{ikk}(t) | + | s_{ikk}(t) - z_i(t) | + | z_i(t) - z(t) |
\]

\[< 3 \cdot 2^{-i} + | s_{ikk}(t) - z_i(t) | .
\]

Since by construction \( \lim_{k \to \infty} s_{ikk}(t) = z_i(t) \), it follows that \( \lim_{k \to \infty} s_k(t) = z(t) \) for every \( t \), so that \( z \in L(P) \).

**Theorem 1.** If \( P \) is a norm-closed cone in a real Banach space \( X \), then \( K_X(]X P) \) is norm-closed in \( X^{**} \).

**Proof.** Let \( F \in X^{**} \) be the limit in norm of a sequence \( \{ F_n \} \subset K_X(]X P) \). Thus each \( F_n \) is the \( \omega^* \)-limit of a bounded \( [3, p. 209] \) sequence \( J_XS_n \), where \( S_n = \{ s_{nk} \} \subset P \).

**Case 1.** First suppose that \( X \) is a closed subspace of \( C[0, 1] \). For each \( t \in [0, 1] \) let \( f_t \in X^* \) be defined by \( f_t(s) = s(t) \) for all \( s \in X \). Then for each \( n \) it follows that \( F_n(f_t) = \lim_{k \to \infty} s_{nk}(t) \). If \( z_n \) is defined on \( [0, 1] \) by \( z_n(t) = F_n(f_t) \), it follows that \( z_n \in L(P) \) and moreover, since \( \| f_t \| \leq 1 \) for each \( t \), that \( \| z_n - z_m \| \leq \| F_n - F_m \| \) for all \( n \) and \( m \), so that the function \( z(t) = \lim_{n \to \infty} z_n(t) \) exists and belongs to \( L(P) \) by Lemma 2. Thus there is a bounded sequence \( \{ s_k \} \subset P \) which converges point-wise to \( z \). Since \( X \) is a subspace of \( C[0, 1] \), for each \( f \in X^* \) there is a finite regular signed Borel measure \( \mu_f \) on \( [0, 1] \) such that \( f(x) \)
for every \( x \in X \) [3, p. 397]. Hence

\[
F_n(f) = \lim_{k \to \infty} f(s_{nk}) = \int_0^1 s_n d\mu_f
\]

for every \( n \), and therefore

\[
F(f) = \lim_{n \to \infty} \int_0^1 s_n d\mu_f = \int_0^1 \mu_f = \lim_{n \to \infty} \int_0^1 s_n d\mu_f = \lim_{k \to \infty} f(s_k),
\]

so that \( F \) is the \( \omega^* \)-limit of \( \{ J x s_k \} \).

**Case 2.** If \( X \) is an arbitrary real Banach space, let \( Y \) be the norm-closed subspace and \( Q \) the norm-closed cone in \( X \) generated by \( \{ s_{nk} : n, k \in \omega \} \). Since \( Y \) is separable, there is an equivalence mapping \( E \) from \( Y \) onto a closed subspace \( Z \) of \( C[0,1] \). Since each \( F_n \in K_x(J_x P) \), for each \( n \) an element \( g_n \in Y^{**} \) is unambiguously defined by

\[
G_n(f | Y) = F_n(f) \quad \text{for all } f \in X^*.
\]

The sequence \( \{ G_n \} \) is clearly Cauchy since each \( g \in Y^* \) has an extension \( f \in X^* \) such that \( \| f \| = \| g \| \). Since \( G_n \in K_Y(Q) \), it follows that the sequence \( \{ E^{**} G_n \} \) is a Cauchy sequence in \( K_Z(J_Z E Q) \) converging in norm to \( E^{**} G \). By Case 1 there is a bounded sequence \( \{ z_k \} \subset E Q \) such that \( \{ J x s_k \} \) is \( \omega^* \)-convergent to \( E^{**} G \) in \( Z^{**} \). Finally, it is straightforward to verify that the sequence \( \{ J x E^{-1} z_k \} \subset J x P \) is \( \omega^* \)-convergent to \( F \) in \( X^{**} \).

3. For each \( t \in [0,2] \) and \( q \leq \omega \) let \( f_{t,q} \in C[0,2] \) be defined by

\[
f_{t,q}(t) = \max \left[ 0, 1 - 2^q \right] t - t_q \quad \text{for all } t \in [0,2].
\]

If \( p, i, \) and \( j \) are positive integers, let \( s_{pi} = 2^{-p} i \) and \( t_j = 2 - 2^{-j} \). Now if \( a \geq 1 \) let \( X^a \) be the norm-closed subspace of \( C[0,2] \) generated by the set \( G^a = \{ x_{pq}^a : p, q \in \omega \} \), where

\[
x_{pq}^a(t) = \max \left[ \max_{1 \leq j \leq p} f_{s_{pi}, p+q}(t), \max_{p \leq j < p+q} f_{s_{pi}, j}(t) \right].
\]

**Lemma 3.** If \( P^a \) is the norm-closed cone in \( X^a \) generated by \( \{ x_{pq}^a : p, q \in \omega \} \), then the function \( x_a \) defined by

\[
x_a(t) = \lim_{p \in \omega} \lim_{q \in \omega} x_{pq}^a(t)
\]

is an element of \( L(L(P^a)) \) with \( \| x_a \| = 1 \), but if \( \{ y^h \} \) is a bounded sequence in \( L(P^a) \) converging pointwise to \( x_a \), then \( \lim \sup_{h \in \omega} \| y^h \| \geq a \).

**Proof.** It is trivial that \( x_a \in L(L(P)) \) and \( \| x_a \| = 1 \). Indeed, \( x_a \) is
the characteristic function of the set \( D = \{ s_{\pi i} : p \in \omega, i < 2^p \} \). If \( \{ y^k \} \) is an arbitrary bounded sequence in \( L(P^a) \) converging pointwise to \( x_0 \), then since the set of all finite linear combinations of the \( x_{pq}^a \) with non-negative coefficients is dense in \( P^a \), each \( y^k \) is the pointwise limit of a bounded sequence \( \{ y^{hk} \}_{k=1}^\infty \), where each \( y^{hk} \) has the form

\[
y^{hk}(t) = \sum_{p \geq 1; q \geq 1} a^{hk}_{pq} x_{pq}(t),
\]

where each \( a^{hk}_{pq} \) is non-negative and for each pair \((h, k)\) only a finite number of the \( a^{hk}_{pq} \) are positive. Without changing the value of \( \limsup_{k \to \omega} \| y^k \| \), it may be assumed that \( y^{hk}(\frac{1}{2}) = 1 \) for all \((h, k)\).

Let \( \epsilon > 0 \) be given. For each \( H \in \omega \) let

\[
S_H = \{ t \in [0, 1] : t \in D, y^h(t) < \epsilon \text{ for all } h \geq H \}.
\]

Since \( \bigcup_{H \in \omega} S_H \) is of the second category in \([0, 1]\), there exists an \( H \) such that \( S_H \) is dense in a closed interval \( I \). Choose an arbitrary \( s_{p_0 i_0} \in D \cap (\text{int } I) \) with \( i_0 \) odd; then there exists \( H_0 \geq H \) such that \( y^h(s_{p_0 i_0}) > 1 - \epsilon \) for all \( h \geq H_0 \). Since there exist points of \( S_H \) between \( s_{p_0 i_0} \) and every \( s_{pi} \) such that \( p < p_0 \), it is clear that for every \( h \geq H_0 \), there exists \( K_h \) such that for every \( k \geq K_h \),

\[
\sum_{p \geq 1; q \geq 1} a^{hk}_{pq} y^k(s_{p_0 i_0}) - \sum_{p < p_0; q \geq 1} a^{hk}_{pq} s_{pq}(s_{p_0 i_0}) > (1 - \epsilon) - 2\epsilon.
\]

Now fix \( h \geq H_0 \). Since \( y^h \) is a Baire function of the first class, there exists a point \( \sigma H \in I \) such that \( y^h|I \) is continuous at \( \sigma H \) \([2, p. 143]\); hence there is a closed interval \( J \subset I \) such that \( y^h(t) < 2\epsilon \) for all \( t \in J \). Choose \( s_{p_1 i_1} \in D \cap (\text{int } J) \) with \( p_1 > p_0 \); then there exists \( K' \geq K_h \) such that for every \( k \geq K' \),

\[
\sum_{p \geq 1; q \geq 1} a^{hk}_{pq} y^k(s_{p_1 i_1}) < 2\epsilon.
\]

Next, by (3.2), for every \( t \in [0, 1] \) and every \( k \),

\[
y^{hk}(t) \geq \sum_{p \geq 1; q \geq 1} a^{hk}_{pq} (1 - 2^{p+q} | t - s_{p_0 i_0} | ) \geq \sum_{p \geq 1; q \geq 1} a^{hk}_{pq} (1 - 2^{p_1} | t - s_{p_0 i_0} | ).
\]

There exists \( t_0 \in S_H \) such that \( | t_0 - s_{p_0 i_0} | < 2^{-(2p_1+1)} \), and there exists \( K_h' \geq K_h ' \) such that \( y^{hk}(t_0) < \epsilon \) whenever \( k \geq K_h ' \). Hence, by (3.8) with \( t = t_0 \),
for all \( k \geq K'' \). Therefore, by (3.6), (3.7), and (3.9),

\[
(3.10) \quad \sum_{p \in S \subset P; \xi \in P} a_{pq} > 1 - 7\varepsilon
\]

for all \( k \geq K'' \), so that \( \|y_n\| \geq \lim_{k \to \infty} y^{(k)}(t_p) \geq a(1 - 7\varepsilon) \). Since \( \varepsilon \) may be an arbitrarily small positive number, it follows that \( \lim_{s_{k \to \infty}} \|y_n\| \geq a \).

**Theorem 2.** There exist a norm-closed subspace \( X \) of \( \mathfrak{C}[0, 1] \) and a norm-closed cone \( P \subset X \) such that \( K_X(K_X(J_XP)) \) is not norm-closed in \( X^{**} \).

**Proof.** For each \( r \in \omega \) and each real function \( x \) defined on \([0, 2]\) let \( E_r x \) be the function defined on \([0, 1]\) by

\[
(3.11) \quad (E_r x)(t) = \begin{cases} x(2^{r+1}[t - 2^{-r}]) & \text{if } 2^{-r} \leq t \leq 2^{1-r}, \\ 0 & \text{for all other } t. \end{cases}
\]

Recalling the notation of Lemma 3 with \( a = 2^r \), observe that \( E_r X^{**} \) is an equivalence mapping onto a subspace of \( \mathfrak{C}[0, 1] \), since \( x(0) = x(2) = 0 \) for all \( x \in X^{**} \). Let \( X \) be the closed subspace and \( P \) the closed cone in \( \mathfrak{C}[0, 1] \) generated by \( \cup_{r \in \omega} \{ E_r P^{**} \} \). For each \( r \) it is clear that \( E_r x \in L(L(P)) \) and hence that each \( F_r \in X^{**} \) can be unambiguously defined by

\[
(3.12) \quad F_r(f) = \int_0^1 (E_r x_0) d\mu_f \quad \text{for all } f \in X^*;
\]

then \( F_r \in K_X(K_X(J_XP)) \) and \( \|F_r\| = 1 \). Now let \( F = \sum_{r \geq 1} 2^{-r} F_r \); thus \( F \) belongs to the closure of \( K_X(K_X(J_XP)) \) in the norm topology.

Suppose \( F \in K_X(K_X(J_XP)) \). Then there is a bounded sequence \( \{ G_h \} \subset K_X(J_XP) \) whose \( w^\ast \)-limit is \( F \). For each pair \((r, h) \) let \( z_{r h} \) be the function on \([0, 2]\) defined by

\[
(3.13) \quad z_{r h}(t) = G_h(f_{2^{-r} t + 2^{-r} - 1}) \quad \text{for } t \in [0, 2],
\]

where \( f_u \in X^* \) is defined by \( f_u(x) = x(u) \) for \( x \in X \) and real \( u \). Then \( \{ z_{r h} \}_{r = 1}^{\infty} \) is a bounded sequence in \( L(P^{**}) \) which converges pointwise on \([0, 2]\) to \( 2^{-r} z_{r 0} \). By Lemma 3, \( \lim_{s_{k \to \infty}} \|z_{r h}\| \geq 2^r \), and hence \( \lim_{s_{k \to \infty}} \|G_h\| \geq 2^r \). Since \( r \) is an arbitrary positive integer, the sequence \( \{ G_h \} \) is unbounded, which gives a contradiction. Thus \( F \notin K_X(K_X(J_XP)) \) and the theorem is proved.
Remark. The author has not been able to determine whether \( K_x(K_x(J_xX)) \) can fail to be norm-closed in \( X^{**} \).

References


Florida State University