THE RADIUS OF CONVEXITY FOR STARLIKE FUNCTIONS OF ORDER 1/2

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1. Introduction. Suppose that \( f(z) = z + a_2z^2 + \cdots \) is analytic for \(|z| < 1\). The condition

\[
\text{Re} \left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > 0 \quad \text{for} \quad |z| < r \tag{1}
\]

is necessary and sufficient for \( f(z) \) to be univalent and convex for \(|z| < r \) \([3, \text{p. 105, problem 108}]\). The condition

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{for} \quad |z| < r \tag{2}
\]

is necessary and sufficient for \( f(z) \) to be univalent and starlike for \(|z| < r \) \([3, \text{p. 105, problem 109}]\). Since each convex function is starlike a function which is convex in \(|z| < 1\) satisfies (2) with \( r = 1 \). In fact, for functions convex in \(|z| < 1\) (2) can be improved to

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{1}{2} \quad \text{for} \quad |z| < 1 \tag{3}
\]

[2; 6]. In each of these references it is also shown that if \( f(z) \) is convex in \(|z| < 1\) then

\[
\text{Re} \left\{ \frac{f(z)}{z} \right\} > \frac{1}{2} \quad \text{for} \quad |z| < 1. \tag{4}
\]

As Strohhäcker has pointed out neither (3) nor (4) are sufficient for the convexity of \( f(z) \) in \(|z| < 1\). Indeed, a function may satisfy (4) without being univalent for \(|z| < 1\).

In this paper we determine the largest circle \(|z| < r\) such that each function satisfying (3) is convex in \(|z| < r\). Also, we find the radius of univalence (and starlikeness) for functions satisfying (4).

Some properties of functions subject to (3) have been discussed in \([5]\). These functions are a particular case of the so-called starlike functions of order \( p \) which are defined by the condition

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > p \quad \text{for} \quad |z| < 1
\]

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2. **Lemma 1.** The function \( g(z) \) is analytic for \( |z| < 1 \) and satisfies \( g(0) = 1 \) and \( \Re g(z) > 1/2 \) for \( |z| < 1 \) if and only if \( g(z) = 1/(1+z\phi(z)) \) where \( \phi(z) \) is analytic and satisfies \( |\phi(z)| \leq 1 \) for \( |z| < 1 \).

**Proof.** To prove one part of this lemma suppose that \( g(z) = 1+b_1z+\cdots \) is analytic and \( \Re g(z) > 1/2 \) for \( |z| < 1 \). Let \( h(z) = 2g(z) - 1 = 1 + 2b_1z + \cdots \), \( \Re h(z) > 0 \) for \( |z| < 1 \). Let \( k(z) = (1-h(z))/(1+h(z)) \). Then, \( k(z) \) is analytic and \( |k(z)| < 1 \) for \( |z| < 1 \), and \( k(0) = 0 \). Therefore, \( k(z) = z\phi(z) \), where \( \phi(z) \) is analytic and satisfies \( |\phi(z)| \leq 1 \) for \( |z| < 1 \). Solving for \( g(z) \) gives \( g(z) = 1/(1+z\phi(z)) \).

The converse follows readily from the fact that \( w = 1/(1+z) \) maps \( |z| < 1 \) onto \( \Re w > 1/2 \).

The following lemma gives a way of constructing starlike functions of order 1/2 in terms of bounded functions.

**Lemma 2.** The function \( f(z) \) is analytic for \( |z| < 1 \) and satisfies \( f(0) = 0 \), \( f'(0) = 1 \), and \( \Re \{zf'(z)/f(z)\} > 1/2 \) for \( |z| < 1 \) if and only if

\[
f(z) = z \exp \left\{ - \int_0^z \frac{\phi(\sigma)}{1 + \sigma \phi(\sigma)} \, d\sigma \right\}
\]

where \( \phi(z) \) is analytic and \( |\phi(z)| \leq 1 \) for \( |z| < 1 \).

**Proof.** To prove one-half of this lemma suppose that \( f(z) = z + a_2z^2 + \cdots \) is analytic and \( \Re \{zf'(z)/f(z)\} > 1/2 \) for \( |z| < 1 \). The function \( f(z)/z \) is analytic and does not vanish for \( |z| < 1 \). Applying Lemma 1 to the function \( zf'(z)/f(z) \) gives \( zf'(z)/f(z) = 1/(1+z\phi(z)) \) where \( \phi(z) \) is analytic and satisfies \( |\phi(z)| \leq 1 \) for \( |z| < 1 \).

\[
\frac{d(f(z)/z)}{dz} = \frac{f'(z)}{f(z)} - \frac{1}{z} = -\frac{\phi(z)}{1+z\phi(z)}
\]

\[
\log \frac{f(z)}{z} = -\int_0^z \frac{\phi(\sigma)}{1 + \sigma \phi(\sigma)} \, d\sigma
\]

\[
f(z) = z \exp \left\{ - \int_0^z \frac{\phi(\sigma)}{1 + \phi(\sigma)} \, d\sigma \right\}.
\]

Conversely, suppose that \( f(z) \) has such a representation. The analyticity of \( f(z) \) for \( |z| < 1 \) and \( f(0) = 0 \) follow directly. Also, from this representation we obtain
\[ f'(z) = \frac{1}{1 + z\phi(z)} \exp \left\{ - \int_0^z \frac{\phi(\sigma)}{1 + \sigma \phi(\sigma)} \, d\sigma \right\}, \]
\[ \frac{zf'(z)}{f(z)} = \frac{1}{1 + z\phi(z)}. \]

Then, according to Lemma 1, \( \text{Re}\{zf'(z)/f(z)\} > 1/2 \) for \( |z| < 1 \). Also, \( f'(0) = 1 \).

**Theorem 1.** Suppose that \( f(z) = z + a_2z^2 + \cdots \) is analytic and satisfies \( \text{Re}\{zf'(z)/f(z)\} > 1/2 \) for \( |z| < 1 \). Then, \( f(z) \) maps \( |z| < (2^{3/4} - 3)^{1/2} \) onto a convex domain.

**Proof.** Applying Lemma 1 to the function \( zf'(z)/f(z) \) gives
\[ \frac{zf'(z)}{f(z)} = \frac{1}{1 + z\phi(z)} \]
where \( \phi(z) \) is analytic and satisfies \( |\phi(z)| \leq 1 \) for \( |z| < 1 \). For such functions we have
\[ |\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \quad [1, \text{p. 18}]. \]
Differentiating (5) and multiplying through by \( f(z)/f'(z) \) yields
\[ \frac{zf''(z)}{f'(z)} + 1 - \frac{zf'(z)}{f(z)} = - \frac{f(z)}{f'(z)} \frac{\phi(z) + z\phi'(z)}{(1 + z\phi(z))^2}. \]
Using (5) again gives
\[ \frac{zf''(z)}{f'(z)} + 1 = \frac{1 - z\phi(z) - z^2\phi'(z)}{1 + z\phi(z)}. \]
From this it follows that the condition \( (1) \text{Re}\{(zf''(z)/f'(z)) + 1\} > 0 \)
is equivalent to
\[ \text{Re}\{(1 - z\phi(z) - z^2\phi'(z))(1 + z\phi(z))^*\} > 0,^1 \]
\[ \text{Re}\{1 - |z|^2|\phi(z)|^2 - z^2\phi'(z)(1 + z\phi(z))^*\} > 0, \]
\[ \text{Re}\{z^2\phi'(z)(1 + z\phi(z))^*\} < 1 - |z|^2|\phi(z)|^2. \]
From (6) we get
\[ \text{Re}\{z^2\phi'(z)(1 + z\phi(z))^*\} \leq |z|^2|\phi'(z)| (1 + |z| |\phi(z)|) \]
\[ \leq \frac{|z|^2}{1 - |z|^2} (1 - |\phi(z)|^2)(1 + |z| |\phi(z)|). \]

^1 Asterisks denote the conjugate of a complex number.
Therefore, (7) will be satisfied if
\[ \frac{|z|^2}{1 - |z|^2} (1 - |\phi(z)|^2) (1 + |z| |\phi(z)|) < 1 - |z|^2 |\phi(z)|^2. \]
This inequality is equivalent to
\[ (8) \quad 2|z|^2 + |z| (1 - |z|^2) |\phi(z)| - |z|^2 |\phi(z)|^2 < 1. \]
To discuss (8) let us consider the function \( p(x) = 2a^2 + a(1 - a^2)x - a^2 x^2 \) for \( 0 \leq x \leq 1 \), where \( 0 < a < 1 \) (\( a = |z| \), \( x = |\phi(z)| \)). If \( 0 < a \leq 2^{1/2} - 1 \) then \( p(x) \) is increasing and has the maximum value \( q(a) = a + a^2 - a^3 \). The function \( q(a) \) also increases (for \( 0 < a < 1 \)) and \( q(2^{1/2} - 1) < 1 \). This implies that \( p(x) < 1 \) for all \( x, 0 \leq x \leq 1 \), if \( 0 < a \leq 2^{1/2} - 1 \).

Now suppose that \( a > 2^{1/2} - 1 \). Then \( p(x) \) achieves the maximum value of \( (1/4)(1 + 6a^2 + a^4) \) at \( x = (1 - a^2)/2a \). Since the inequality \( (1/4)(1 + 6a^2 + a^4) < 1 \) is equivalent to \( a < (2(3^{1/2} - 3)^{1/2}) \) we can infer that \( p(x) < 1 \) for all \( x, 0 \leq x \leq 1 \), if \( 2^{1/2} - 1 < a < (2(3^{1/2} - 3)^{1/2}) \).

We have shown: \( p(x) < 1 \) for all \( x, 0 \leq x \leq 1 \), if \( a < (2(3^{1/2} - 3)^{1/2}) \). Therefore, (8) is satisfied for every function \( \phi(z) \) where \( |\phi(z)| \leq 1 \) if \( |z| < (2(3^{1/2} - 3)^{1/2}) \). Since (8) implies (1) this proves that \( f(z) \) is convex in \( |z| < (2(3^{1/2} - 3)^{1/2}) = 0.68 \cdots \).

Let us show that \( f(z) \) need not be convex in the circle \( |z| < r \) if \( r > (2(3^{1/2} - 3)^{1/2}) \). Let \( a = (2(3^{1/2} - 3)^{1/2}, b = a/3^{1/2}, \phi(z) = (z - b)/(1 - bz) \). Since \( 0 < b < 1 \) this function maps \( |z| < 1 \) onto \( |\phi| < 1 \). A direct computation shows that \( 1 - z\phi(z) - z^2 \phi'(z) \) vanishes at \( z = a \). A function \( f(z) \) can be constructed which satisfies the hypotheses of the theorem and for which \( zf''(z)/f'(z) = 1/(1 + z\phi(z)) \). Then \( (zf''(z)/f'(z)) + 1 \) vanishes at \( z = a \) since
\[ \frac{zf''(z)}{f'(z)} + 1 = \frac{1 - z\phi(z) - z^2 \phi'(z)}{1 + z\phi(z)}. \]
Therefore, \( f(z) \) is convex in no circle \( |z| < r \) with \( r > a \).

**Theorem 2.** If \( f(z) = z + az^2 + \cdots \) is analytic and satisfies \( \text{Re}\{f(z)/z\} > 1/2 \) for \( |z| < 1 \) then \( f(z) \) is univalent and starlike for \( |z| < 1/2^{1/2} \).

**Proof.** Applying Lemma 1 to \( f(z)/z \) gives
\[ f(z) = \frac{z}{1 + z\phi(z)} \]
where \( \phi(z) \) is analytic and satisfies \( |\phi(z)| \leq 1 \) for \( |z| < 1 \). Thus,
\[ |\phi'(z)| \leq \frac{(1 - |\phi(z)|^2)/(1 - |z|^2). \] From (9) it follows that
\[ \frac{zf'(z)}{f(z)} = \frac{1 - z^2\phi'(z)}{1 + z\phi(z)}. \]

Let \( z \) be a complex number with \( |z| < 1/2^{1/2} \). Then,
\[ |z^2\phi'(z)| \leq \frac{|z|^2}{1 - |z|^2} (1 - |\phi(z)|^2) < 1 - |\phi(z)|^2 \leq 1. \]

The number \( 1 - z^2\phi'(z) \) lies in a circle with center at \( w = 1 \) and with radius less than \( 1 - |\phi(z)|^2 \). Thus, \( |\arg (1 - z^2\phi'(z))| < \arcsin (1 - |\phi(z)|^2) \) for \( |z| < 1/2^{1/2} \). Similarly, \( |\arg (1 + z\phi(z))| \leq \arcsin |z\phi(z)| \) for all \( z, |z| < 1 \).

Therefore, if \( |z| < 1/2^{1/2} \)
\[ |\arg \frac{zf'(z)}{f(z)}| \leq |\arg (1 - z^2\phi'(z))| + |\arg (1 + z\phi(z))| \]
\[ < \arcsin (1 - |\phi(z)|^2) + \arcsin \frac{|\phi(z)|}{2^{1/2}} \]
\[ = \arcsin \left( \left(1 - \frac{|\phi(z)|^2}{2} \right)^{1/2} \right) \]
\[ \leq \arcsin 1 \]
\[ = \frac{\pi}{2}. \]

This shows that \( \text{Re}\{|zf'(z)/f(z)| > 0 \) for \( |z| < 1/2^{1/2} \). Therefore, \( f(z) \) is univalent and starlike for \( |z| < 1/2^{1/2} \).

To show that this result is a best-possible let us consider the function
\[ f(z) = \frac{z}{1 + z\phi(z)} \text{ where } \phi(z) = \frac{z - a}{1 - az} \text{ and } a = \frac{1}{2^{1/2}}. \]

Then, \( |\phi(z)| < 1 \) for \( |z| < 1 \), and \( f(z) \) satisfies the hypotheses of this theorem. However, \( f(z) \) is not univalent in \( |z| < r \) if \( r > a \) since a short computation shows that \( f'(a) = 0 \).

**References**


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INVARIANT SUBSPACES IN $L^1$

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1. Denote by $L^1$ and $L^2$ the spaces of summable and square summable functions on the circle group and by $H^1$ and $H^2$ the subspaces of $L^1$ and $L^2$ consisting of those functions whose Fourier coefficients vanish for negative indices. A subspace $M$ of $L^1$ or of $L^2$ is said to be invariant if

$$\chi \cdot M \subseteq M$$

and to be doubly invariant if also

$$\bar{\chi} \cdot M \subseteq M$$

where $\chi$ is the character

$$\chi(e^{i\theta}) = e^{i\theta}.$$

$H^1$ and $H^2$ are closed subspaces which are invariant but not doubly invariant.

Invariant subspaces on the circle were originally studied by Beurling [1] who showed that the closed invariant subspaces contained in $H^2$ are of the form $\phi \cdot H^2$ where $\phi$ is a function in $H^2$ which has modulus one a.e. Such functions are called inner functions. Rudin and de Leeuw [3, p. 476] have shown that the closed invariant subspaces in $H^1$ have the same structure as those in $H^2$. That is to say, they are of the form $\phi \cdot H^1$ where again $\phi$ is an inner function. The arguments given in [1; 3] depend to a considerable extent on the function theory of the spaces $H^1$ and $H^2$.

Recently, Helson and Lowdenslager [2] using Hilbert space methods have given a very elegant and simple proof of Beurling's theorem.

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