A SEQUENCE OF INTEGERS RELATED TO THE BESSEL FUNCTIONS

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Let \( j_{r}, r \) denote the zeros of \( \pi^{-1}J_{1}(z) \), where \( J_{1}(z) \) is the Bessel function of the first kind, and put

\[
\sigma_{2n}(v) = \sum_{r=1}^{\infty} (j_{r}, r)^{-2n} \quad (n = 1, 2, 3, \ldots).
\]

Properties of \( \sigma_{2n}(v) \) have been discussed in a recent paper by Kishore [2]. We remark that \( \sigma_{2n}(v) \) is a rational function of \( v \) with rational coefficients; the first twelve functions have been computed by Lehmer [3].

For \( v = \pm \frac{1}{2} \), \( \sigma_{2n}(v) \) is expressible in terms of the numbers of Bernoulli and Genocchi by means of the following formulas:

\[
\sigma_{2n}(\pm \frac{1}{2}) = (-1)^{n-1} \frac{2^{2n-1}}{(2n)!} B_{2n}
\]

\[
\sigma_{2n}(\pm \frac{1}{2}) = (-1)^{n} \frac{2^{2n-2}}{(2n)!} G_{2n}
\]

where

\[ B^{n} = (B + 1)^{n} \quad (n \neq 1), \quad G_{n} = 2(1 - 2^{n}) B_{n}. \]

In view of the known arithmetic properties of these and related numbers, it is of some interest to look for arithmetic properties of \( \sigma_{2n}(v) \) for other values of \( v \). In the present note we consider the case \( v = 0 \). Elsewhere [1] the writer has discussed the coefficients of \((J_{0}(v))^{-1}\). It will be convenient to define

\[
\kappa_{r} = 2^{r} r! (r - 1)! \sigma_{2r}(0) \quad (r \geq 1).
\]

Thus the formulas [2, (14), (22)]

\[
\sum_{r=1}^{n} (-1)^{r} 2^{r} (r!)^{2} \binom{n}{r} \binom{\nu + n}{r} \sigma_{2r}(\nu) + n = 0,
\]

\[
(\nu + n) \sigma_{2n}(\nu) = \sum_{r=1}^{n-1} \sigma_{2r}(\nu) \sigma_{2n-2r}(\nu)
\]

reduce to

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Since \( a_1 = 1 \) we find, using either (3) or (4), that
\[
\begin{align*}
a_2 &= 1, \\
a_3 &= 2^2, \\
a_4 &= 3 \cdot 11, \\
a_5 &= 2^4 \cdot 3 \cdot 19, \\
a_6 &= 2^4 \cdot 5 \cdot 11 \cdot 43, \\
a_7 &= 2^4 \cdot 3 \cdot 5^3 \cdot 229, \\
a_8 &= 3 \cdot 5 \cdot 7 \cdot 167 \cdot 607.
\end{align*}
\]

It is evident from (4) that the \( a_n \) are positive integers. If \( n = p \), a prime, it follows from (3) that
\[
a_p \equiv 1 \pmod{p}.
\]

This result can be extended. We recall that if \([4]\)
\[
n = n_0 + n_1 p + n_2 p^2 + \cdots \quad (0 \leq n_j < p),
\]
\[
r = r_0 + r_1 p + r_2 p^2 + \cdots \quad (0 \leq r_j < p)
\]
then
\[
\binom{n}{r} \equiv \binom{n_0}{r_0} \binom{n_1}{r_1} \binom{n_2}{r_2} \cdots \pmod{p}.
\]

In particular (6) implies
\[
\binom{m p}{r p} \equiv \binom{m}{r}, \quad \binom{m p - 1}{r p - 1} \equiv \binom{m - 1}{r - 1} \pmod{p}
\]
and
\[
\binom{m p}{r} \equiv 0 \pmod{p} \quad (p \nmid r).
\]

Thus if \( n = mp \), (3) becomes
\[
\sum_{r=1}^{n} (-1)^r \binom{m}{r} \binom{m - 1}{r - 1} a_{rp} + 1 \equiv 0 \pmod{p}.
\]
Comparing (7) with (3) it is evident that

\[ a_{mp} \equiv a_m \pmod{p} \]  

and somewhat more generally

\[ a_{mp^r} \equiv a_m \pmod{p} \quad (r = 1, 2, 3, \ldots). \]

We show next that

\[ a_n \equiv 0 \pmod{p} \quad (\phi < n < 2\phi). \]

For \( n = \phi + 1 \), this is obvious from (4). Assuming that (10) holds up to and including the value \( n \), (4) implies

\[ a_{n+1} = \sum_{r=n-p+1}^{p} \binom{n}{r} \binom{n}{r-1} a_r a_{n-r+1} \pmod{p}. \]

If \( n + 1 = \phi + m \), where \( 1 \leq m < \phi \), then by (6)

\[ \binom{n}{r} = \binom{\phi + m - 1}{r} \equiv \binom{m - 1}{r} \equiv 0 \pmod{p} \quad (m \leq r < \phi), \]

\[ \binom{n}{\phi - 1} = \binom{\phi + m - 1}{\phi - 1} \equiv \binom{m - 1}{\phi - 1} \equiv 0 \pmod{p}. \]

Thus (11) reduces to \( a_{n+1} \equiv 0 \pmod{p} \). This completes the proof of (10).

It is now easy to prove the more general congruence

\[ a_n \equiv 0 \pmod{p} \quad (n > \phi, \phi \nmid n) \]

by induction. Indeed if

\[ n + 1 = k\phi + m, \quad 1 \leq m < \phi, \]

then by (6) and the inductive hypothesis

\[ a_{n+1} \equiv 2 \binom{n}{m} \binom{n}{m-1} a_m a_{k\phi} \pmod{p}; \]

but by (6)

\[ \binom{n}{m} = \binom{k\phi + m - 1}{m} \equiv \binom{m - 1}{m} \equiv 0, \]

so that \( a_{n+1} \equiv 0 \pmod{p} \).

We remark that

\[ a_n \equiv 0 \pmod{n - 1} \quad (n > 1). \]
Indeed if we put

\[ a_n = (n - 1)b_n \quad (n > 1), \]

(4) becomes for \( n > 1 \)

\[ nb_{n+1} = 2n(n-1)b_n + \sum_{r=2}^{n-1} \binom{n}{r} \binom{n-1}{r-1} (n-r)(r-1)b_rb_{n-r+1} \]

\[ = 2n(n-1)b_n + n^2 \sum_{r=2}^{n-1} \binom{n-1}{r} \binom{n-1}{r-2} b_rb_{n-r+1}, \]

so that

\[ b_{n+1} = 2(n-1)b_n + n \sum_{r=2}^{n-1} \binom{n-1}{r} \binom{n-1}{r-2} b_rb_{n-r+1} \quad (n > 1). \]

Since \( b_2 = 1 \) it is evident from (14) that \( b_n \) is integral for all \( n \geq 2 \). This proves (13).

Returning to (4) it is clear that

\[ a_{p+1} \equiv 2pa_p \pmod{p^2}. \]

Combining this with (5) we get

(15) \[ a_{p+1} \equiv 2p \pmod{p^2}. \]

Similarly we have for \( p > 2 \)

\[ a_{p+2} \equiv 2a_{p+1} + 2\left( \binom{p+2}{2} (p+1)a_p \pmod{p^2}. \right. \]

Using (5) and (15) this reduces to

(16) \[ a_{p+2} \equiv 5p \pmod{p^2}. \]

In the same way if \( p > 3 \) we get

\[ a_{p+3} \equiv 2a_{p+2} + 2\left( \binom{p+2}{2} (p+2)a_{p+1} \right. \]

\[ + 8\left( \binom{p+2}{3} \binom{p+2}{2} a_p \pmod{p^2}. \right) \]

which reduces to

(17) \[ a_{p+3} \equiv \frac{q_2}{3} p \pmod{p^2} \quad (p > 3). \]

To get a general result of this kind we put
(18) \[ a_{p+n} \equiv p c_n \pmod{p} \quad (0 < n < p), \]
so that by (10) \( c_n \) is integral. Now by (4)
\[
a_{p+n} = \sum_{r=1}^{p+n-1} \binom{p+n-1}{r} \binom{p+n-1}{r-1} a_r a_{p+n-r}
\equiv 2 \sum_{r=1}^{n} \binom{p+n-1}{r} \binom{p+n-1}{r-1} a_r a_{p+n-r} \pmod{p^2}.
\]
Making use of (18) and (6) this becomes
(19) \[ c_n^{(p)} = 2 \sum_{r=1}^{n-1} \binom{n-1}{r} \binom{n-1}{r-1} a_r c_{n-r} + \frac{2}{n} a_n \pmod{p} \quad (1 < n < p). \]
By means of (19) the \( c_n^{(p)} \) can be computed. However, it is simpler to define a single sequence \( \{c_n\} \) by means of
(20) \[ c_n = 2 \sum_{r=1}^{n-1} \binom{n-1}{r} \binom{n-1}{r-1} a_r c_{n-r} + \frac{2}{n} a_n \quad (n > 1) \]
with \( c_1 = 2 \). We evidently have
(21) \[ c_n^{(p)} \equiv c_n \pmod{p} \quad (0 < n < p). \]
The \( c_n \) as defined by (20) are not integral. If we put
(22) \[ c_n' = n c_n \quad (n \geq 1), \quad c_0 = 1, \]
then (20) becomes
(23) \[ c_n' = 2 \sum_{r=1}^{n} \binom{n}{r} \binom{n-1}{r-1} a_r c_{n-r} \quad (n \geq 1); \]
the \( c_n' \) are therefore integral. Moreover it follows from (23) that
\[
\sum_{n=1}^{\infty} \frac{c_n' \left( \frac{x}{2} \right)^{2n}}{n!(n-1)!} = 2 \sum_{r=1}^{\infty} \frac{a_r \left( \frac{x}{2} \right)^{2n}}{r!(r-1)!} \sum_{n=0}^{\infty} \frac{c_n' \left( \frac{x}{2} \right)^{2n}}{n!n!}. \]
If we put
\[
C(x) = \sum_{n=0}^{\infty} \frac{c_n' \left( \frac{x}{2} \right)^{2n}}{n!n!}, \quad A(x) = \sum_{r=1}^{\infty} \frac{a_r \left( \frac{x}{2} \right)^{2n}}{r!(r-1)!},
\]
then
\[ \frac{x}{2} C'(x) = \sum_{n=1}^{\infty} \frac{c_n}{n!(n-1)!} \left( \frac{x}{2} \right)^{2n} \]

so that

\[ \frac{x}{2} C'(x) = 2A(x)C(x). \]

But [2]

\[ A(x) = \sum_{r=1}^{\infty} a_r(0)x^r = -\frac{1}{2} x \frac{J'_0(x)}{J_0(x)}. \]

It follows that

\[ \frac{C'(x)}{C(x)} = -\frac{J'_0(x)}{J_0(x)}, \]

which yields

(24) \[ C(x) = (J_0(x))^{-1}. \]

We have accordingly found a simple generating function for the \( c_n \).

It follows from (13) that if \( n \equiv 1 \pmod{p^k} \) then \( a_n \equiv 1 \pmod{p^k} \).

Using (4) and (8) it is not difficult to show that if \( n = mp^k + 1 \) then

(25) \[ a_n \equiv m p^k a_m \pmod{p^{k+1}}. \]

We shall now show that if

(26) \[ n = n_s p^s + n_{s+1} p^{s+1} + \cdots + n_t p^t \quad (0 \leq n_j < p) \]

and \( n_s \geq 1, n_t \geq 1 \), then

(27) \[ a_n \equiv 0 \pmod{p^{t-s}}. \]

The proof is by induction on \( n \). We use (4) with \( n \) replaced by \( n-1 \).

Let \( 1 \leq r < n \) and put

\[ r = r_s p^s + r_{s+1} p^{s+1} + \cdots + r_t p^t \quad (0 \leq r_j < p), \]

\[ m = n - r = m_s p^s + m_{s+1} p^{s+1} + \cdots + m_t p^t \quad (0 \leq m_j < p). \]

Clearly either \( s' \leq s \) or \( s'' \leq s \); also \( t' \leq t \) and \( t'' \leq t \). If \( t' = t'' = t \) there is evidently nothing to prove.

(i) \( t' = t, t'' < t \). By the inductive hypothesis

\[ a_r \equiv 0 \pmod{p^{t-s}}, \quad a_{n-r} \equiv 0 \pmod{p^{t'-s'}}. \]
so that
\[ a_r a_{n-r} \equiv 0 \pmod{p^{t'' + t'' - s' - s''}}. \]
If \( s' \leq t'' \) it follows (since \( t'' \geq \max (s', s'') \)) that \( t' + t'' - s' - s'' \geq t - s \).
If however \( s' > t'' \) we examine the binomial coefficient \( C_{n-1,n} \).

We recall that if
\[
\begin{align*}
 n &= n_0 + n_1 p + \cdots + n_k p^k \\
 S(n) &= n_0 + n_1 + \cdots + n_k,
\end{align*}
\]
then \( n! \) is divisible by exactly \( p^s \), where
\[ (p - 1)^s = n - S(n). \]
It follows that \( \binom{n - 1}{r} \) is divisible by \( p^{t - t''} \). Since \((t' + t'' - s' - s'') + (s' - s'') = t + t'' - 2s \geq t - s \) we get
\[ \binom{n - 1}{r} a_r a_{n-r} \equiv 0 \pmod{p^{t-s}}. \]

(ii) \( t' < t, t'' < t \). We may suppose that \( t'' \leq t' = t - 1 \). Also it is clear that \( s' \leq t'' \). Then
\[ t' + t'' - s' - s'' \geq t - s - 1; \]
indeed if \( s \min (s', s'') \) we get
\[ t' + t'' - s' - s'' \geq t - s. \]
Thus only the case \( s = \min (s', s'') \) requires further examination. With the present hypothesis we evidently have
\[ \binom{n}{r} \equiv 0 \pmod{p}; \]
but when \( s = \min (s', s'') \) then either \( n \) and \( r \) or \( n \) and \( n-r \) are divisible by the same power of \( p \). It follows that either
\[ \binom{n - 1}{r} \quad \text{or} \quad \binom{n - 1}{r - 1} \]
is divisible by \( p \). Consequently
\[ \binom{n - 1}{r} \binom{n - 1}{r - 1} a_r a_{n-r} \equiv 0 \pmod{p^{t-s}}. \]
This completes the proof of (26).

Summary. The sequence of positive integers \( \{a_n\} \) defined by (2) —or alternatively by (3) or (4)—have the following properties.
1. \(a_{mp} = a_m, \quad a_p \equiv 1 \pmod{p}.

2. \(a_n \equiv 0 \pmod{p} \quad (n > p, \ p + n).

3. \(a_n \equiv 0 \pmod{n - 1} \quad (n > 1).

4. \(a_{p+n} \equiv c_n p \pmod{p^n} \quad (1 \leq n < p),

where the \(c_n\) are defined by

\[
1 + \sum_{n=1}^{\infty} \frac{c_n \binom{x}{2^n}}{(n - 1)!(n - 1)!} = (J_0(x))^{-1};
\]

moreover \(nc_n\) is integral.

5. If \(n = mp^k + 1\) then

\[a_n \equiv mp^ka_m \pmod{p^{k+1}}.\]

6. If \(p^k \mid n, \ p^{k+1} \nmid n < p^{k+1}\) then

\[a_n \equiv 0 \pmod{p^{k-1}}.\]

The following values of \(a_n\) were computed by R. Carlitz in the Duke University Computing Laboratory.

\[
a_9 = 2^4 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 1607
\]

\[
a_{10} = 2^4 \cdot 3^2 \cdot 7 \cdot 199 \cdot 328981
\]

\[
a_{11} = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 83 \cdot 3000553
\]

\[
a_{12} = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 2389 \cdot 4153159
\]

\[
a_{13} = 2^2 \cdot 3^2 \cdot 5 \cdot 7^4 \cdot 11 \cdot 29 \cdot 97 \cdot 139 \cdot 1663
\]

\[
a_{14} = 2^4 \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 37 \cdot 107 \cdot 1283 \cdot 5952613
\]

\[
a_{15} = 2^4 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 23 \cdot 563 \cdot 797 \cdot 227966279
\]

\*

\[
a_{16} = 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 41 \cdot 2390700514417253
\]

\[
a_{17} = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 461 \cdot 1342361 \cdot 33327739
\]

\[
a_{18} = 2^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 79 \cdot 199729 \cdot 139135943558279
\]

\[
a_{19} = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot a_{19}'
\]

\[
a_{20} = 2^4 \cdot 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 1109 \cdot a_{20}'
\]

\[
a_{21} = 2^4 \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 47 \cdot a_{21}'
\]

\[
a_{22} = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot a_{22}'
\]

\[
a_{23} = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot a_{23}'
\]

\[
a_{24} = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot a_{24}'
\]
a_{26} = 2^6 \cdot 3^4 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot a'_{26}

a_{27} = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 23 \cdot 79 \cdot a'_{27}

a_{28} = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot a'_{28}

a_{29} = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 43 \cdot a'_{29}

a_{30} = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot a'_{30}

The numbers $a'_{19}, \ldots, a'_{30}$ have not been factored completely but at any rate have no prime divisors <10^4. The number $a'_{30}$ has 47 digits.

References


Duke University