UNIQUE FACTORIZATION FOR POLYNOMIALS

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Suppose $R$ is a unique factorization domain and $R[x]$ is not. Let $p$ be a polynomial in $R[x]$ minimal in degree with respect to the property that $p$ has two essentially different prime (irreducible polynomial) factorizations. We may assume

$$p = f_1 f_2 \cdots f_r = g_1 g_2 \cdots g_s,$$

where all $f_i$ and $g_i$ are primes with $f_i \neq u g_i$ for any unit $u$ and where

$$m = \deg f_1 \geq \deg f_2 \geq \cdots \geq \deg f_r,$$

$$n = \deg g_1 \geq \deg g_2 \geq \cdots \geq \deg g_s,$$

and $n \geq m > 0$. Call $a$, $b$ the coefficients of $x^n$, $x^m$ in $f_1$, $g_1$ respectively, and let

$$q = a p - b f_1 x^{n-m} g_2 \cdots g_s.$$

On one hand

$$q = a f_1 f_2 \cdots f_r - b f_1 x^{n-m} g_2 \cdots g_s,$$

$$= f_1 (a f_2 \cdots f_r - b x^{n-m} g_2 \cdots g_s),$$

while on the other hand

$$q = a g_1 g_2 \cdots g_s - b f_1 x^{n-m} g_2 \cdots g_s,$$

$$= (a g_1 - b f_1 x^{n-m}) g_2 \cdots g_s.$$

From this we see that if $q = 0$, then $a g_1 = b f_1 x^{n-m}$. If, however, $q \neq 0$, then note $\deg (a g_1 - b f_1 x^{n-m}) < \deg g_i$ and hence $\deg q < \deg p$, so that $q$ must have a unique factorization into primes, some of which are $g_2, \cdots, g_s$ and $f_1$. But then $f_1$ must be a factor of $(a g_1 - b f_1 x^{n-m})$ and hence also of $a g_1$. Therefore, in either case, $a g_1 = f_1 h$ for some polynomial $h$, so that, since $f_1$ is prime and $\deg f_1 > 0$ and $\deg a = 0$, then $a$ is a factor of $h$; hence, $h = a h'$ for some polynomial $h'$, and $a g_1 = f_1 a h'$, or $g_1 = f_1 h'$. This shows that $f_1$ is a factor of $g_1$, which is impossible by our assumptions. Hence, $R[x]$ must indeed be a unique factorization domain. This generalization to polynomials of a well-known direct proof for unique factorization in the natural numbers arose out of an incidental remark by Professor C. W. Curtis in one of his courses at the University of Wisconsin.

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Received by the editors February 7, 1962 and, in revised form, January 7, 1963.

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