UNIQUE FACTORIZATION FOR POLYNOMIALS

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Suppose R is a unique factorization domain and $R[x]$ is not. Let p be a polynomial in $R[x]$ minimal in degree with respect to the property that p has two essentially different prime (irreducible polynomial) factorizations. We may assume $p=f_1f_2\cdots f_r=g_1g_2\cdots g_s$, where all f_i and g_j are primes with $f_i\neq ug_j$ for any unit u and where

$$m = \deg f_1 \geqq \deg f_2 \geqq \cdots \geqq \deg f_r,$$
$$n = \deg g_1 \geqq \deg g_2 \geqq \cdots \geqq \deg g_s.$$

and $n\geqq m>0$. Call a, b the coefficients of x^m, x^n in f_1, g_1 respectively, and let

$$q = ap - bf_1x^{n-m}g_2\cdots g_s$$

On one hand

$$\begin{aligned} q &= af_1f_2\cdots f_r - bf_1x^{n-m}g_2\cdots g_s \\ &= f_1(af_2\cdots f_r - bx^{n-m}g_2\cdots g_s), \end{aligned}$$

while on the other hand

$$\begin{aligned} q &= ag_1g_2\cdots g_s - bf_1x^{n-m}g_2\cdots g_s \\ &= (ag_1 - bf_1x^{n-m})g_2\cdots g_s. \end{aligned}$$

From this we see that if $q=0$, then $ag_1=bf_1x^{n-m}$. If, however, $q\neq 0$, then note $\deg(ag_1-bf_1x^{n-m})<\deg g_1$ and hence $\deg q<\deg p$, so that q must have a unique factorization into primes, some of which are $g_2,\cdots,g_s$ and f_1. But then f_1 must be a factor of $(ag_1-bf_1x^{n-m})$ and hence also of ag_1. Therefore, in either case, $ag_1=f_1h$ for some polynomial h, so that, since f_1 is prime and $\deg f_1>0$ and $\deg a=0$, then a is a factor of h; hence, $h=ah'$ for some polynomial h', and $ag_1=f_1ah'$, or $g_1=f_1h'$. This shows that f_1 is a factor of g_1, which is impossible by our assumptions. Hence, $R[x]$ must indeed be a unique factorization domain. This generalization to polynomials of a well-known direct proof for unique factorization in the natural numbers arose out of an incidental remark by Professor C. W. Curtis in one of his courses at the University of Wisconsin.

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Received by the editors February 7, 1962 and, in revised form, January 7, 1963.