1. If $G$ is a Lie group, then the group $\text{Aut}(G)$ of all continuous automorphisms of $G$ has a natural Lie group structure. This gives the semi-direct product $A(G) = G \cdot \text{Aut}(G)$ the structure of a Lie group. When $G$ is a vector group $\mathbb{R}^n$, $A(G)$ is the ordinary affine group $A(n)$. Following L. Auslander [1] we will refer to $A(G)$ as the affine group of $G$, and regard it as a transformation group on $G$ by $(g, \alpha): h \mapsto g \cdot \alpha(h)$ where $g, h \in G$ and $\alpha \in \text{Aut}(G)$; in the case of a vector group, this is the usual action on $A(n)$ on $\mathbb{R}^n$.

If $B$ is a compact subgroup of $A(n)$, then it is well known that $B$ has a fixed point on $\mathbb{R}^n$, i.e., that there is a point $x \in \mathbb{R}^n$ such that $b(x) = x$ for every $b \in B$. For $A(n)$ is contained in the general linear group $\text{GL}(n+1, \mathbb{R})$ in the usual fashion, and $B$ (being compact) must be conjugate to a subgroup of the orthogonal group $O(n+1)$. This conjugation can be done leaving fixed the $(n+1, n+1)$-place matrix entries, and is thus possible by an element of $A(n)$. This done, the translation parts of elements of $B$ must be zero, proving the assertion.

L. Auslander [1] has extended this theorem to compact abelian subgroups of $A(G)$ when $G$ is connected, simply connected and nilpotent. We will give a further extension.

**Theorem.** Let $G$ be a connected Lie group and let $S$ be the identity component of the radical of $G$. Then the following conditions are equivalent:

1. If $B$ is a compact subgroup of the affine group $A(G)$, then $G$ has an element $x$ such that $b(x) = x$ for every $b \in B$.

2. Every compact subgroup of $A(G)$ is conjugate to a subgroup of $\text{Aut}(G)$.

3. $G$ has no nontrivial compact subgroup.

4. $G$ is homeomorphic to Euclidean space.

5. $S$ is simply connected and $G/S$ is a direct product of copies of the universal covering group of the real special linear group $\text{SL}(2, \mathbb{R})$.

6. $G$ is simply connected, and every simple analytic subgroup of $G$ is a 3-dimensional noncompact group.

Equivalence of (3) and (4) is contained in the Cartan-Iwasawa theorem [3, Theorem 13]. Equivalence of (4) and (5) follows from
Chevalley's theorem on the topology of solvable groups [2], the fact that the universal covering of $\text{SL}(2, \mathbb{R})$ is the only simple Lie group homeomorphic to Euclidean space, and the global Levi-Whitehead decomposition of $G$. It is not difficult to see that (5) is equivalent to (6) and it is clear that (1) is equivalent to (2). Finally, a nontrivial compact subgroup of $G$ is a compact subgroup of $\text{A}(G)$ which is not conjugate to a subgroup of $\text{Aut}(G)$; thus (2) implies (3). The proof of the Theorem is now reduced to the proof that (3) implies (2).

2. Suppose that $G$ has no nontrivial compact subgroup. Then $G$ is simply connected, and it follows that $\text{A}(G)$ has only finitely many connected components because $\text{Aut}(G)$ is isomorphic to the group $\text{Aut}(G)$ of automorphisms of the Lie algebra $\mathfrak{g}$ of $G$, and $\text{Aut}(G)$ is a real algebraic matrix group. Thus $\text{A}(G)$ has only finitely many connected components. The Cartan-Iwasawa theorem [3, Theorem 13] is valid for Lie groups with only finitely many components; thus $\text{A}(G)$ has maximal compact subgroups and, if $K$ is one of them, every compact subgroup of $\text{A}(G)$ is conjugate to a subgroup of $K$. The proof that (3) implies (2) is now reduced to the proof that $\text{Aut}(G)$ contains a maximal compact subgroup of $\text{A}(G)$.

Let $K \subseteq \text{A}(G) \subseteq \text{A}(G)$ be a maximal compact subgroup of $\text{A}(G)$; we will prove that $K$ is a maximal compact subgroup of $\text{A}(G)$. Let $K'$ be a maximal compact subgroup of $\text{A}(G)$ with $K \subseteq K'$; we must prove $K = K'$. It is easily seen that $K$ meets every component of $\text{A}(G)$; it follows that we need only prove that $K$ and $K'$ have the same identity component. Again because $K \subseteq K'$, it suffices to show that $\dim K = \dim K'$. Let $f: \text{A}(G) \rightarrow \text{A}(G)$ be the canonical homomorphism $(g, \alpha) \rightarrow \alpha$ with kernel $G$. $K \cap G$ and $K' \cap G$ are compact subgroups of $G$ and thus are trivial by hypothesis. Furthermore $K = f(K) = f(K')$ because $K$ is a maximal compact subgroup of $\text{A}(G)$, and because $f(K)$ is contained in the compact subgroup $f(K')$. This gives $\dim K = \dim f(K') = \dim K'$ which proves the Theorem.

3. It is worth remarking that the main part of the Theorem—the equivalence of (1), (2) and (3)—can be proved in the same way when $G$ is assumed to have only finitely many connected components.

References


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