The purpose of this note is to point out a simple generalization of the inequality
\[(z_1z_2 \cdots z_n)^{1/n} \leq \frac{1}{n} (z_1 + \cdots + z_n)\]
of arithmetic and geometric means, which will hold when the arguments of the complex numbers \(z_1, \cdots, z_n\) are suitably restricted. We shall apply the resulting inequality to the roots of polynomial equations, obtaining first a quantitative form of the Gauss-Lucas theorem, and then some relationships between the coefficients of a polynomial and the size of a sector containing its roots.

1. The inequality. The basic result is

**Theorem 1.** Suppose
\[|\arg z_i| \leq \psi < \frac{\pi}{2}, \quad i = 1, 2, \cdots, n.\]

Then
\[|z_1z_2 \cdots z_n|^{1/n} < \left(\sec \psi\right) \frac{1}{n} |z_1 + z_2 + \cdots + z_n|\]

unless \(n\) is even and \(z_1 = \cdots = z_{n/2} = \bar{z}_{(n/2)+1} = \cdots = \bar{z}_n = re^{i\psi}\), in which case equality holds.

**Proof.** We have
\[|z_1 + z_2 + \cdots + z_n| \geq |\Re(z_1 + \cdots + z_n)|\]
\[= (|z_1| \cos \phi_1 + |z_2| \cos \phi_2 + \cdots
\[\quad + |z_n| \cos \phi_n)\]
\[\geq \cos \psi (|z_1| + \cdots + |z_n|)\]
\[\geq n \cos \psi (|z_1| + |z_2| + \cdots + |z_n|)^{1/n}\]

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263
as claimed. All signs of equality hold only when

(a) \( \text{Im} (z_1 + \cdots + z_n) = 0 \)
(b) \( \cos \phi_i = \cos \psi \quad (i = 1, 2, \cdots, n) \)
(c) \( |z_1| = |z_2| = \cdots = |z_n| \)

which imply the configuration stated in the theorem. For odd \( n \) the constant \( \sec \psi \) is only asymptotically best possible.

2. Application to polynomials. Let

\[
P(z) = a_0 + a_1 z + \cdots + a_n z^n = a_n (z - z_1) \cdots (z - z_n)
\]

be given and let \( K \) denote the convex hull of the zeros \( z_1, \cdots, z_n \) of \( P(z) \). Let \( z \) be outside \( K \), and suppose that, from \( z \), \( K \) subtends an angle \( 2\psi \). Then the spread in the arguments of the numbers

\[
\frac{1}{z - z_1}, \ldots, \frac{1}{z - z_n}
\]

is at most \( 2\psi \), and from Theorem 1,

\[
\left| \frac{1}{(z - z_1)} \cdots \frac{1}{(z - z_n)} \right|^{1/n} \leq (\sec \psi) \frac{1}{n} \left| \sum_{i=1}^{n} \frac{1}{z - z_i} \right|.
\]

But this is just the assertion that

\[
\left| \frac{a_n}{P(z)} \right|^{1/n} \leq \frac{\sec \psi}{n} \left| \frac{P'(z)}{P(z)} \right|,
\]

and we have proved

**Theorem 2.** If \( z \) is a point from which the convex hull of the zeros of the polynomial \( P(z) \) of degree \( n \) subtends an angle \( 2\psi \leq \pi \), then

\[
|P'(z)| \geq n |a_n|^{1/n} (\cos \psi) |P(z)|^{1-(1/n)}.
\]

**Corollary 1.** The zeros of \( P'(z) \) lie in the convex hull of the zeros of \( P(z) \) (Gauss-Lucas).

**Corollary 2.** If the zeros of \( P(z) \) lie in the unit circle, then we have for \( |z| > 1 \),

\[
|P'(z)| \geq n |a_n|^{1/n} \sqrt{1 - \frac{1}{|z|^2}} |P(z)|^{1-(1/n)}.
\]
Theorem 3. The zeros of the polynomial
\[ P(z) = a_0 + a_1 z + \cdots + a_n z^n, \]
are not contained in any sector of central angle less than
\[ 2 \cos^{-1} \left\{ \min_{0 \leq k \leq n-1} \left| \frac{a_n}{n a_n} \right| a_n \left( \frac{n}{k} \right)^{1/n-k} \right\}. \]

Proof. Suppose the zeros of \( P(z) \) lie in a sector of angle \( 2\psi < \pi \).
From Theorem 1,
\[ \left| \frac{a_0}{a_n} \right|^{1/n} \geq \frac{\sec \psi}{n} \left| a_{n-1} \right|, \]
or
\[ \sec \psi \geq n \left| a_n \right|^{1-(1/n)} \left| a_0 \right|^{1/n} \left| a_{n-1} \right|^{-1}. \]
Applying this result to
\[ P^{(k)}(z) = \sum_{\nu=0}^{n-k} \frac{(\nu+k)!}{\nu!} a_{\nu+k} z^\nu, \]
which, by Corollary 1 also satisfies the hypotheses, we find
\[ \sec \psi \geq \left| \frac{na_n}{a_{n-1}} \right| \frac{a_n}{a_k} \left( \frac{n}{k} \right)^{-1/n-k} \quad (k = 0, 1, \ldots, n-1), \]
and the result follows.

Theorem 4. Under the hypotheses of Theorem 2, let \( \rho \) denote the distance from \( z \) to the center of gravity of the zeros of \( P(z) \). Then
\[ \left| P(z) \right| \leq \left| a_n \right| (\rho \sec \psi)^n. \]

Proof. Apply Theorem 1 to the numbers \( z - z_1, \ldots, z - z_n \).

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