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THE HEBREW UNIVERSITY

WHITTAKER'S CONSTANT FOR LACUNARY ENTIRE FUNCTIONS

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1. **Introduction.** Let

$$f(z) = \sum_{\nu=0}^{\infty} \frac{b_\nu}{\nu!} z^\nu \tag{1}$$

be an entire function of exponential type $\tau < \infty$. We are concerned here with two problems which are closely related to the determination of Whittaker's constant, that is to say, with theorems to the effect that if $f(z)$ and each of its derivatives have some zeros in the unit circle then τ cannot be too small.

DEFINITION 1. The constant W_p is the largest number W for which the following assertion is true: "Let the coefficients b_ν of $f(z)$ vanish except for values of ν in the arithmetic progression $q, q+p, q+2p, \cdots$. If $f(z), f'(z), \cdots$ each have a zero in $|z|<1$, and if $\tau < W$, then $f(z) \equiv 0$."

One sees, by considering $f^{(q)}(z)$ that W_p is independent of q. W_1 is Whittaker's constant, whose value is unknown [1]. The case $p=2$ has also been investigated [2; 3].

DEFINITION 2. The constant ω_p is the largest number ω for which the following assertion is true: "Let $f(z), f'(z), \cdots$ each have at least p zeros in $|z|<1$. If $\tau < \omega$, then $f(z) \equiv 0$."

Again $\omega_1 = W_1$ is Whittaker's constant. Erdös-Rényi [6, equation (15)] have shown that

$$\omega_p \geqq p/e \qquad (p = 1, 2, \cdots). \tag{2}$$

We shall give a somewhat better bound.

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2. **The constants W_p.** Our main result is

THEOREM 1. *The asymptotic expansions, for $p\to\infty$, of W_p and of $(p!)^{1/p}$, in terms of powers of p and* $\log p$, *are identical. In particular,*

$$W_p = \frac{p}{e} + \frac{\log p}{2e} + \log\sqrt{2\pi} + o(1) \qquad (p\to\infty). \tag{3}$$

To prove this, let us define

$$h(z) = \sum_{m=1}^{\infty} \frac{z^{mp}}{(mp)!} \tag{4}$$

and then state

LEMMA 1. *The constant W_p is not greater than the modulus of the root of smallest modulus of $h(z)=-1$, nor is it smaller than the unique positive real root of $h(z)=+1$.*

In fact, we have

$$\begin{aligned} f^{(q+np)}(a_n) = 0 &= \sum_{\nu=0}^{\infty} \frac{b_{q+\nu+np}}{\nu!} a_n^{\nu} \\ &= \sum_{m=0}^{\infty} \frac{b_{q+(n+m)p}}{(mp)!} a_n^{mp} \end{aligned}$$

for some $|a_n| \leqq 1$ $(n=0, 1, \cdots)$. Hence

$$|b_{q+np}| \leqq \sum_{m=1}^{\infty} \frac{|b_{q+(n+m)p}|}{(mp)!} \qquad (n = 0, 1, \cdots). \tag{5}$$

Suppose $h(\tau)<1$. Choose ϵ so that $h(\tau+\epsilon)<1$. Since $f(z)$ is of type τ,

$$|b_{q+np}| \leqq A(\tau+\epsilon)^{q+np} \qquad (A = A(\epsilon)).$$

Inductively, suppose it has been shown that

$$|b_{q+np}| \leqq A(\tau+\epsilon)^{q+np}\{h(\tau+\epsilon)\}^r \tag{6}$$

for some r. Then substitution of (6) into (5) yields (6) with r replaced by $r+1$, hence (6) is true for every r. Since $h(\tau+\epsilon)<1$, $f(z)\equiv 0$ and the second half of Lemma 1 is proved. Next if λ is the modulus of the zero of smallest modulus of $1+h(z)$ then put

$$g(z) = 1 + h(\lambda z).$$

Then $g^{(p)}(z)=\lambda^p g(z)$, and $g(z)$ is of type λ, whence $W_p\leqq\lambda$, completing the proof of the lemma.

Now consider the equation $h(x)=1$, and put $y=x^p/p!$, getting

$$1 = y + y^2 \frac{p!^2}{(2p)!} + y^3 \frac{p!^3}{(3p)!} + \cdots. \tag{7}$$

Since the root we seek is surely in $0<y<2$, we have

$$\left| y^3 \frac{p!^3}{(3p)!} + \cdots \right| \leqq y^2 \frac{(p!)^2}{(2p)!} \frac{3}{3^p - 2}$$

for such y. Putting $(p!)^2(2p)!^{-1} = \delta$, we find for the positive root of (7)

$$y = 1 - y^2\delta(1 + O(3^{-p}))$$

and since $\delta \sim (\pi p)^{1/2} 2^{-2p-1}$

$$y = 1 - \delta + O(p \cdot 12^{-p}).$$

Hence the positive root of $h(x) = 1$ is of the form[1]

$$(p!)^{1/p} \left\{ 1 - \frac{\delta}{p} + O(12^{-p}) \right\} \qquad (p \to \infty). \tag{8}$$

For the equation $h(z) = -1$ we use the fact that the function

$$h(z^{1/p}) + 1 = \sum_{m=0}^{\infty} \frac{z^m}{(mp)!}$$

has only real zeros ([4], 5, 160). Being of order $1/p$ it has the form

$$1 + h(z^{1/p}) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{\alpha_n} \right) \qquad (0 < \alpha_1 \leqq \alpha_2 \leqq \cdots). \tag{9}$$

We may now use the method of Euler exactly as in [5, p. 500] and find that

$$\alpha_1 \leqq \left\{ \frac{p!}{1 - \dfrac{2(p!)^2}{(2p)!}} \right\}^{1/p}. \tag{10}$$

Theorem 1 now follows from (8), (10) and Lemma 1.

3. **The constant ω_p.** In terms of the Bessel function

$$I_0(x) = \sum_{n=0}^{\infty} \frac{\left(\dfrac{x}{2}\right)^{2n}}{(n!)^2} \tag{11}$$

[1] I am indebted to Professor de Bruijn for this estimate. My own method gave only the first two terms in (3).

we shall prove, by a similar argument,

THEOREM 2. *We have*

$$\omega_p \geqq \frac{1}{2} \max_{x>0} \left\{ \frac{x}{[I_0(x)]^{1/2p}} \right\} \qquad (p = 1, 2, 3, \cdots)$$

$$\geqq \frac{p}{e} + \frac{\log p}{4e} + O(1) \qquad (p \to \infty). \tag{12}$$

PROOF. If $f^{(k)}(z)$ has p zeros in $|z| < 1$, then by Jensen's theorem,

$$p \log \rho \leqq \frac{1}{2\pi} \int_0^{2\pi} \log \left| f^{(k)}(\rho e^{i\theta}) \right| d\theta - \log \left| f^{(k)}(0) \right| \qquad (\rho > 1). \tag{13}$$

Instead of estimating the integral with the maximum modulus, we can do somewhat better with the well-known inequality of the geometric mean and r.m.s.,

$$\exp \left\{ \frac{1}{b-a} \int_a^b g(x)dx \right\} \leqq \left\{ \frac{1}{b-a} \int_a^b g^2(x)dx \right\}^{1/2}. \tag{14}$$

Indeed, exponentiating (13), using (14) and Parseval's identity we get

$$\rho^p \leqq \frac{1}{|b_k|} \left\{ \sum_{\nu=0}^{\infty} \frac{|b_{\nu+k}|^2}{(\nu!)^2} \rho^{2\nu} \right\}^{1/2} \qquad (\rho > 1). \tag{15}$$

Just as in the step (5), (6), the assumption

$$|b_k| \leqq A(\tau + \epsilon)^k \qquad (k = 0, 1, 2, \cdots) \tag{16}$$

in (15) leads to

$$|b_k| \leqq A(\tau + \epsilon)^k \left\{ \frac{I_0(2\rho(\tau + \epsilon))^{1/2}}{\rho^p} \right\} \qquad (\rho > 1). \tag{17}$$

The obvious induction shows that if the quantity in braces in (17) is less than unity, then $f(z) \equiv 0$. Hence, for all $\rho > 1$

$$I_0(2\rho\tau)^{1/2} \geqq \rho^p$$

or

$$\tau \geqq \frac{1}{2} \max_{\rho>1} \frac{I_0^{-1}(\rho^{2p})}{\rho} = \frac{1}{2} \max_{x>0} \left\{ x I_0(x)^{-1/2p} \right\}.$$

The asymptotic relation shown in (12) follows at once from choosing $x=2p$ and using well-known asymptotic formulas for $I_0(x)$. The table below shows in the first row, p, in the second, the constant on the right side of (12) and in the third p/e.

1	2	3
0.6897	1.100	1.493
0.368	0.736	1.104

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