ON THE BEHAVIOUR OF THE FOURIER COEFFICIENTS

SULAXANA KUMARI

1.1. Let \( f(\theta) \) be a function integrable in the sense of Lebesgue over \((-\pi, \pi)\) and periodic outside this range with period \(2\pi\). Let the Fourier series associated with \( f(\theta) \) be

\[
(1.1.1) \quad f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n(\theta).
\]

Then the series conjugate to the above Fourier series is

\[
(1.1.2) \quad \sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta) = \sum_{n=1}^{\infty} B_n(\theta).
\]

We write

\[
\phi(t) = \phi_0(t) = \frac{1}{2} \{f(x + t) + f(x - t) - 2s\}; \\
\psi(t) = \frac{1}{2} \{f(x + t) - f(x - t)\}; \\
\theta(t) = \psi(t) - l; \\
\Phi_0(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - u)^{\beta-1} \phi(u) du, \quad \beta > 0; \\
\Phi_\beta(t) = \frac{d}{dt} \Phi_{\beta+1}(t), \quad -1 < \beta < 0; \\
\phi_\beta(t) = \Gamma(\beta + 1)t^{-\beta} \Phi_\beta(t), \quad \beta > -1,
\]

and define \( \Psi_\alpha(t), \psi_\alpha(t), \Theta_\alpha(t) \) and \( \theta_\alpha(t) \) in a similar way.

1.2. In 1930 it was proved by Bosanquet [3] that if \( \phi_\alpha(t) \to 0 \), as \( t \to 0 \), for \( \alpha > 0 \), then the Fourier series of \( f(\theta) \), at \( \theta = x \), is summable \((C, \alpha + \delta)\) for every \( \delta > 0 \). This result however breaks down for \( \delta = 0 \) [15]. For the case \( \alpha = 0 \), Hardy and Littlewood [8] obtained the convergence of the Fourier series (1.1.1), at \( \theta = x \), under a set of stricter conditions, namely

(i) \( \phi(t) = o\left(\frac{1}{\log 1/t}\right) \), as \( t \to 0 \),

and

(ii) \( A_n = O(n^{-\delta}) \), for some \( \delta > 0 \).

They have also shown that their result breaks down if condition (ii)
is omitted. They however replaced condition (i) by a weaker condition

\[(i)' \quad \int_0^t |\phi(u)| \, du = o\left(\frac{t}{\log 1/t}\right), \quad \text{as} \quad t \to 0 \quad [9].\]

In 1947 Wang [16] obtained the summability \((C, \alpha)\) of the Fourier series for every \(\alpha > 0\), whenever

\[(iii) \quad \phi_\alpha(t) = o\left(\frac{1}{\log 1/t}\right), \quad \text{as} \quad t \to 0,\]

the order condition on the Fourier coefficients being dropped. It is easy to see that the above condition can also be replaced by the weaker condition

\[(iii)' \quad \int_0^t |\phi_\alpha(u)| \, du = o\left(\frac{t}{\log 1/t}\right), \quad \text{as} \quad t \to 0.\]

Recently Mohanty and Nanda [12] have proved that if \(\theta(t) = o(1/\log 1/t), \) as \(t \to 0,\) and \(a_n, b_n = O(n^{-\delta}), \delta > 0,\) then \(nB_n(x) \to (2l/\pi)(C, 1).\) Further, putting \(I = 0\) and supposing that the conjugate function

\[\frac{1}{\pi} \int_{-\delta}^{\delta} \psi(t) \cot \frac{t}{2} \, dt\]

exists as a Cauchy integral at the origin, they obtained the convergence of the conjugate series, a result due to Hardy and Littlewood [8].

Analogous results for the \((C, \alpha + 1)\) summability of the sequence \(\{nB_n(x)\}\) and the \((C, \alpha)\) summability of the conjugate series, for \(\alpha > 0,\) have been obtained recently by the authoress [13]. The object of the present paper is to obtain similar results for the case \(-1 < \alpha < 0.\)

Just as with condition (i) Hardy and Littlewood required a stronger order condition on the Fourier coefficients for the convergence of the Fourier series, similarly with the condition (iii)' for \(-1 < \alpha < 0,\) we require a stronger nonlocal condition than is necessary,\(^1\) for the \((C, \alpha)\) summability of the Fourier series, namely \(A_n = O(n^{-\delta}),\) for some \(\delta > 0,\) with a similar remark applying to the summability \((C, \alpha)\) of the conjugate series. It is our conjecture that if we drop \(\delta\) in the\(^1\)

\(^1\) It is proved by Bosanquet and Offord [6] and Bhatnagar [1] that \(A_n = o(n^\alpha)\) or \(B_n = o(n^\alpha)\) whenever the Fourier series or the conjugate series is summable \((C, \alpha),\) for \(-1 < \alpha < 0,\) and these are the only nonlocal conditions needed for such summabilities.
above condition we will fail to obtain summability \((C, \alpha)\) of the Fourier series under condition (iii)'.

We devote §2 for obtaining criteria for summability \((C, \alpha+1)\) of the sequence \(\{nB_n(x)\}\) and for summability \((C, \alpha)\) of the conjugate series, for \(-1<\alpha<0\). In §3 we obtain similar results for the summability of the sequence \(\{nA_n(x)\}\) and of the Fourier series.

I am grateful to Professor B. N. Prasad for his helpful guidance during the preparation of this paper.

2.1. **Theorem 1.** If, for \(-K_0<\alpha<0\),

\[
\Theta_\alpha^*(t) = \int_0^1 |d\Theta_{\alpha+1}(u)| = o\left(\frac{t^{\alpha+1}}{\log 1/t}\right) \quad \text{as } t \to 0,
\]

and

\[B_n = O(n^{\alpha-t}), \quad \delta > 0,\]

then the sequence \(\{nB_n(x)\}\) is summable \((C, \alpha+1)\) to sum \(2l/\pi\).

The following lemmas will be required for the proof of Theorem 1:

**Lemma 1** [1]. If, for \(-K_0<\alpha<1\), a(n, t) denotes the \((C, \alpha)\) mean of the series

\[
-\sum_{n=1}^{\infty} \sin nt
\]

and if

\[
\bar{K}_\alpha(n, t) = -\frac{1}{\pi} \frac{\cos (n; t)}{A_n^2 (2 \sin t/2)^{\alpha+1}},
\]

where \(A_n^2 = \Gamma(n+\alpha+1)/\{\Gamma(n+1)\Gamma(\alpha+1)\}\) and \((n; t) = (n+1/2+\alpha/2)t-\alpha\pi/2\), then for \(0<t<\pi\)

\[
\left|k_\alpha(n, t) - \bar{K}_\alpha(n, t) - \frac{1}{\pi} \cot \frac{t}{2}\right| < An^{-1}t^{-3}.
\]

**Lemma 2.** If \(k_\alpha(n, t)\) and \(\bar{k}_\alpha(n, t)\) denote the \((C, \alpha)\) means of the series

\[
\frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \cos nt
\]

and

\[
\frac{2}{\pi} \sum_{n=1}^{\infty} \sin nt,
\]
respectively, and if \( \Delta u_n \) denotes \( u_n - u_{n-1} \), then

\[
n\Delta \hat{k}^{\alpha+1}(n, t) = -\frac{\alpha + 1}{n + \alpha + 1} \frac{d}{dt} k^{\alpha}(n, t)
\]

(2.1.1)

\[
= (\alpha + 1) [\hat{k}^{\alpha}(n, t) - \hat{k}^{\alpha+1}(n, t)]; \quad [2]
\]

\[
k^{\alpha}(n, t), \hat{k}^{\alpha}(n, t), n\Delta \hat{k}^{\alpha+1}(n, t) = O(n),
\]

(2.1.2) for \( 0 < t < \pi, \alpha > -1 \); [2; 5]

(2.1.3) \[
\hat{k}^{\alpha}(n, t) \quad \geq O(n^{-\alpha t^{-\alpha-1}}), \quad \text{for } nt > 1, -1 < \alpha < 0.
\]

(2.1.4) \[
n\Delta \hat{k}^{\alpha+1}(n, t)\}
\]

\[
\text{Proof of (2.1.3), (2.1.4). From Lemma 1 we have}
\]

\[
\hat{k}^{\alpha}(n, t) = O[\hat{k}^{\alpha}(n, t)] + O(1/n) + O(n^{-\alpha t^{-\alpha}})
\]

\[
= O(n^{-\alpha t^{-\alpha-1}}) + O(1/n), \quad \text{if } nt > 1
\]

\[
= O(n^{-\alpha t^{-\alpha-1}}), \quad \text{if } -1 < \alpha < 0 \quad \text{and } nt > 1.
\]

Also from (2.1.1) and (2.1.3) we have

\[
n\Delta \hat{k}^{\alpha+1}(n, t) = O\{\hat{k}^{\alpha}(n, t)\}
\]

\[
= O(n^{-\alpha t^{-\alpha-1}}),
\]

which gives the required result.

**Lemma 3.** If, for \(-1 < \alpha < 0\),

\[
L(n, u) = \frac{1}{\Gamma(-\alpha)} \int_{u}^{X} (t - u)^{-\alpha-1} n\Delta \hat{k}^{\alpha+1}(n, t)dt,
\]

then

\[
\left| L(n, u) \right| \begin{cases} < A u^{\alpha+1}, & u \leq 1/n, \\ < A u^{-\alpha-1}, & u \geq 1/n, \\ \end{cases}
\]

the results holding uniformly in \( X \).

**Proof of Lemma 3.** We have, if \( u + (1/n) < X \),

\[
L(n, u) = \frac{1}{\Gamma(-\alpha)} \int_{u}^{X} (t - u)^{-\alpha-1} n\Delta \hat{k}^{\alpha+1}(n, t)dt
\]

\[
= \frac{1}{\Gamma(-\alpha)} \left\{ \int_{u}^{u+1/n} + \int_{u+1/n}^{X} \right\}
\]

\[
= L_1 + L_2,
\]

\[\text{If } u + 1/n \geq X, \text{ the integral need not be split up.}\]
say. If \( nu > 1 \), then from \((2.1.4)\) we have
\[
L_1 = O\left[ \int_u^{u+(1/n)} (t - u)^{-a-1}n^{-a-1}dt \right] = O[n^{-a}(1/n)^{-a}n^{-a-1}]
= O[u^{-a-1}].
\]
Also by \((2.1.1)\), \((2.1.3)\) and by the second mean value theorem,
\[
L_2 = -\frac{\alpha + 1}{n + \alpha + 1} \frac{1}{\Gamma(-\alpha)} \int_{u+(1/n)}^\infty (t - u)^{-a-1} \frac{d}{dt} k^\alpha(n, t) dt
= O\left[ n^\alpha \int_{u+(1/n)}^\infty \frac{d}{dt} k^\alpha(n, t) dt \right] = O[n^\alpha n^{-a-1} u^{a-1}]
= O[u^{-a-1}].
\]
Thus for \( u > 1/n \),
\[
| L(n, u) | = O[u^{-a-1}].
\]
Next for the case \( nu \leq 1 \), we have from \((2.1.2)\)
\[
L_1 = O\left[ n \int_u^{u+(1/n)} (t - u)^{-a-1}dt \right] = O[n^{a+1}].
\]
Again from \((2.1.1)\), proceeding as before, we obtain
\[
L_2 = O[n^\alpha \int_{u+(1/n)}^\infty k^\alpha(n, t) dt] = O[n^\alpha n^{a+1}]
= O[n^{a+1}],
\]
which gives finally
\[
L(n, u) = O[n^{a+1}].
\]

**Lemma 4** [4]. If \(-1 < a < 0, \beta > a\) and \(\phi(t)\) is a function such that
(i) \(\Phi_{a+1}(t)\) is a function of bounded variation in an interval \((0, \eta)\) and
(ii) \(\Phi_{a+1}(+0) = 0\), then \(\Phi_{a+1}(t)\) is a Lebesgue integral, \(\Phi_{a+1}(+0) = 0\),
and, for almost all \( t \) in \((0, \eta)\),
\[
\Phi_{\beta}(t) = \frac{1}{\Gamma(\beta - a)} \int_0^t (t - u)^{\beta-a-1} d\Phi_{a+1}(u).
\]

**Lemma 5** [10; 11]. If \(\tau_a(n)\) and \(C_a(n)\) denote the \((C, a)\) means of the sequence \(\{nU_n\}\) and the series \(\sum U_n\), respectively, then for \( a > -1 \),
\[
\tau_a(n) = n\{C_a(n) - C_a(n - 1)\} = nC_a'(n),
\]
\[
\tau_{a+1}(n) = (a + 1)\{C_a(n) - C_{a+1}(n)\}.
\]
2.2. Proof of Theorem 1. We have, since

\[ B_n(x) = \frac{2}{\pi} \int_0^x \psi(t) \sin ndt, \]

\[ r_{\alpha+1}(n) = n\Delta C_{\alpha+1}(n) = \int_0^x \psi(t)n\Delta \bar{\Delta}_{\alpha+1}(n, t)dt, \]

where \( r_{\alpha+1}(n) \) and \( C_{\alpha+1}(n) \) denote the \((C, \alpha+1)\) mean of the sequence \( \{nB_n(x)\} \) and the series \( \sum B_n(x) \), respectively. Now using the fact that if \( \psi(t) = 1 \) in \((0, \pi)\) then the sequence \( \{nB_n(x)\} \) tends to the limit \( 2/\pi \), in Cesàro mean of order \( \delta \), for every \( \delta > 0 \), we have

\[
\frac{2l}{\pi} = \int_0^x \theta(t)n\Delta \bar{\Delta}_{\alpha+1}(n, t)dt
\]

(2.2.1)

\[
= \left\{ \int_0^{n^{-x}} + \int_{n^{-x}}^x \right\} \theta(t)n\Delta \bar{\Delta}_{\alpha+1}(n, t)dt, \quad n < 1,
\]

\[
= J + 1,
\]

say. Now by Lemma 4,

\[
J = \frac{1}{\Gamma(-\alpha)} \int_0^{n^{-x}} n\Delta \bar{\Delta}_{\alpha+1}(n, t) \left\{ \int_0^t (t - u)^{-\alpha-1}d\Theta_{\alpha+1}(u) \right\} dt
\]

\[
= \frac{1}{\Gamma(-\alpha)} \int_0^{n^{-x}} d\Theta_{\alpha+1}(u) \left\{ \int_u^x (t - u)^{-\alpha-1}n\Delta \bar{\Delta}_{\alpha+1}(n, t)dt \right\}
\]

\[
= \int_0^{n^{-x}} d\Theta_{\alpha+1}(u)L(n, u),
\]

where

\[
L(n, u) = \frac{1}{\Gamma(-\alpha)} \int_u^{n^{-x}} (t - u)^{-\alpha-1}n\Delta \bar{\Delta}_{\alpha+1}(n, t)dt.
\]

Or

(2.2.2)

\[
J = \left\{ \int_0^{n^{-1}} + \int_{n^{-1}}^{n^{-x}} \right\} d\Theta_{\alpha+1}(u)L(n, u) = J_1 + J_2,
\]

say. It follows from Lemma 3 and the first condition of Theorem 1, that

\[
J_1 = O \left[ \int_0^{n^{-1}} |d\Theta_{\alpha+1}(u)| \right]
\]

(2.2.3)

\[
= o \left[ \frac{n^{\alpha+1}}{\log 1/u} \right]_{u=1}^{u=n} = o(1),
\]
and

\[
J_2 = O\left[ \int_{n-1}^{n-\tau} u^{-a-1} \, d\Theta_{a+1}(u) \right] \\
= O\left[ \Theta_{a+1}^*(u) u^{-a-1} \right] \\
= o(1) + o\left[ \int_{n-1}^{n-\tau} \frac{1}{u \log 1/u} \, du \right] \\
= o(1) + o\left[ \log \log m \right]^{n-\tau}/u^{-1/n} \\
= o(1).
\] (2.2.4)

Next by (2.1.1),

\[
I = \int_{n-\tau}^{\tau} \frac{\theta(t) n \Delta \tilde{K}^{a+1}(n, t) \, dt}{(\alpha + 1) \int_{n-\tau}^{\tau} \theta(t) \left( \tilde{K}^\alpha(n, t) - \tilde{K}^{a+1}(n, t) \right) \, dt} \\
= (\alpha + 1) \int_{n-\tau}^{\tau} \theta(t) \left( \tilde{K}^\alpha(n, t) - \tilde{K}^{a+1}(n, t) \right) \, dt \\
- (\alpha + 1) \int_{n-\tau}^{\tau} \theta(t) \left( \tilde{K}^{a+1}(n, t) - \tilde{K}^{a+1}(n, t) - \frac{1}{\pi} \cot \frac{t}{2} \right) \, dt \\
+ (\alpha + 1) \int_{n-\tau}^{\tau} \theta(t) \left( K^\alpha(n, t) - K^{a+1}(n, t) \right) \, dt \\
= I_1 + I_2 + I_3,
\] (2.2.5)
say, where

\[
\tilde{K}^\alpha(n, t) = -\frac{2}{\pi} \frac{1}{A_n^\alpha} \frac{\cos (n; t)}{(2 \sin t/2)^{\alpha+1}},
\]

\[
A_n^\alpha = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)\Gamma(\alpha + 1)}
\]

and \((n; t)\) is written for \((n + \frac{3}{2} + \alpha/2) - \alpha \pi/2\).

Now from Lemma 1, writing \(\theta^*(t) = \int_0^t |\theta(u)| \, du = o(t/(\log 1/t))\), we have
\[ I_1 = O\left[ \int_{n^{-1}}^{n^2} \theta(t) n^{-1}t^{-2} dt \right] \]
\[ = O\left[ \left\{ n^{-1}t^{-3} \theta(t) \right\}_{i=n^{-1}}^{n^2} \right] \]
\[ + O\left[ n^{-1} \int_{n^{-1}}^{n^2} \theta(t) t^{-2} dt \right] \]
\[ = o(1) + o\left[ n^{-1} \int_{n^{-1}}^{n^2} \frac{1}{t^2 \log 1/t} dt \right] \]
\[ = o(n^{-1}/\log n) = o(1). \]

It can be shown similarly that
\[ I_2 = o(1). \]

Next we have
\[ I_3 = (\alpha + 1) \int_{n^{-1}}^{n^2} \psi(t) K_\alpha(n, t) dt \]
\[ - (\alpha + 1) \int_{n^{-1}}^{n^2} \psi(t) K_{\alpha+1}(n, t) dt \]
\[ - (\alpha + 1)l \int_{n^{-1}}^{n^2} K_\alpha(n, t) dt \]
\[ + (\alpha + 1)l \int_{n^{-1}}^{n^2} K_{\alpha+1}(n, t) dt \]
\[ = I_{3,1} - I_{3,2} - I_{3,3} + I_{3,4}, \]
say. Now by the second mean value theorem,
\[ I_{3,3} = \frac{2l}{\pi} (\alpha + 1) \int_{n^{-1}}^{n^2} \frac{1}{A^n} \cos \left\{ \frac{(n + (\alpha + 1)/2)t - \alpha/2}{(2 \sin t/2)^{\alpha+1}} \right\} dt \]
\[ = O\left[ n^{-\alpha}(\alpha+1)n^{-1} \right] \]
\[ = o(1). \]

Similarly,
\[ I_{3,4} = o(1). \]

Again
\[ I_{3,1} = (\alpha + 1) \sum_{n=1}^{\infty} \int_{n-\pi}^{n+\pi} B_n(x) \sin \nu t \bar{K}_n(n, t) \, dt \]

\[ = -\frac{2}{\pi} \frac{\alpha + 1}{A_n} \sum_{n=1}^{\infty} \left[ B_n(x) \int_{n-\pi}^{n+\pi} \sin \nu t \cdot \cos \left\{ (n + (\alpha + 1)/2)t - \alpha\pi/2 \right\} \, dt \right] \]

\[ = -\frac{1}{\pi} \frac{\alpha + 1}{A_n} \sum_{n=1}^{\infty} \left\{ B_n(x) \left( 2 \sin \frac{1}{2n} \right)^{\alpha-1} \right. \]

\[ \left. \int_{n-\pi}^{n+\pi} \left[ \sin \left\{ \left( \nu + \frac{\alpha + 1}{2} \right)t - \frac{\alpha\pi}{2} \right\} \right] \, dt \right\}, (n-\pi \leq \delta < \pi) \]

\[ (2.2.11) \]

\[ = O \left[ n^{r(\alpha+1)-\alpha} \sum_{n=1}^{\infty} \frac{B_n(x)}{n + \nu + (\alpha + 1)/2} \right] \]

\[ + O \left[ n^{r(\alpha+1)-\alpha} \sum_{n=1}^{\infty} \frac{B_n(x)}{n - \nu + (\alpha + 1)/2} \right] \]

\[ = O \left[ n^{r(\alpha+1)-\alpha} \sum_{n=1}^{\infty} \frac{\nu^{\alpha-\delta}}{\nu - (n + (\alpha + 1)/2)} \right] \]

\[ = O \left[ n^{r(\alpha+1)-\alpha} \sum_{n=1}^{\infty} \frac{\nu^{\alpha-\delta}}{n + (\alpha + 1)/2 - \nu} \right] \]

\[ + O \left[ n^{r(\alpha+1)-\alpha} \sum_{n=n+1}^{\infty} \frac{\nu^{\alpha-\delta}}{\nu - (n + (\alpha + 1)/2)} \right] = P_1 + P_2, \]

say. Now

\[ P_1 = O \left[ n^{r(\alpha+1)-\alpha} \sum_{1 \leq \nu \leq n/2} \frac{\nu^{\alpha-\delta}}{n + (\alpha + 1)/2 - \nu} \right] \]

\[ + O \left[ n^{r(\alpha+1)-\alpha} \sum_{n/2 < \nu \leq n} \frac{\nu^{\alpha-\delta}}{n + (\alpha + 1)/2 - \nu} \right] \]

\[ (2.2.12) \]

\[ = O \left[ n^{r(\alpha+1)-\alpha} \sum_{1 \leq \nu \leq n/2} \nu^{\alpha-\delta} \right] \]

\[ + O \left[ n^{r(\alpha+1)-\alpha} \sum_{n/2 < \nu \leq n} \frac{1}{n + (\alpha + 1)/2 - \nu} \right] \]

\[ = O[n^{r(\alpha+1)-\alpha-1+a+1-\delta}] + O[n^{r(\alpha+1)-\delta} \log n] = o(1), \]

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
choosing $r < \delta/(a+1)$.

Again breaking $P_2$ in two parts, we have, proceeding as before,

$$P_2 = O\left[ n^{r(a+1)-a} \sum_{n+1 \leq \delta \leq 2n} \left\{ \frac{n^{a-\delta}}{\nu - (n + (a + 1)/2)} \right\} \right]$$

(2.2.13)

$$+ O\left[ n^{r(a+1)-a} \sum_{\nu \geq 2n} \left\{ \frac{n^{a-\delta}}{\nu - (n + (a + 1)/2)} \right\} \right]$$

$$= O[n^{r(a+1)-\delta}] + O[n^{r(a+1)-\delta} \log n]$$

$$= o(1).$$

It follows from (2.2.11), · · · , (2.2.13), that

(2.2.14) $I_{3,1} = o(1).$.

With exactly similar arguments it can be shown that

(2.2.15) $I_{3,2} = o(1).$.

Collecting all the results (2.2.1), · · · , (2.2.15), we obtain

$$\tau_{a+1}(n) = \frac{2l}{\pi} + o(1).$$

This completes the proof of Theorem 1.

2.3. Lemma 6 [9]. If $\sum u_n$ is summable $(A)$, then a necessary and sufficient condition that it should be summable $(C, k)$, for $k > -1$, is that the sequence $\{nu_n\}$ is summable $(C, k+1)$ to the value zero.

As direct application of Theorem 1 and Lemma 6 and the fact that the existence of the conjugate function

$$\frac{1}{\pi} \int_0^\pi \psi(t) \cot \frac{t}{2} dt$$

is a necessary condition for the negative order summability of the conjugate series [1] and that it implies summability $A$ of the conjugate series, we obtain the following theorem:

**Theorem 2.** If, for $-1 < \alpha < 0$,

$$\int_0^t |d\Psi_{a+1}(u)| = o\left( \frac{n^{a+1}}{\log 1/t} \right),$$

as $t \to 0$,

and

$$B_n = O(n^{a-\delta}), \quad \delta > 0,$$
then the necessary and sufficient condition that the conjugate series 
(1.1.2), at \( \theta = x \), may be summable \((C, \alpha)\), \(-1 < \alpha < 0\), to the value

\[
\frac{1}{\pi} \int_{-\theta}^{\theta} \psi(t) \cot \frac{t}{2} dt,
\]

is that the above integral exists as a Cauchy integral at the origin.

2.4. With the help of Lemma 5, the following theorem can be established.3

**Theorem 3.** If, for \(-K < \alpha < 0\),

\[
\int_{0}^{1} \left| d\Theta_{n+1}(u) \right| = o \left( \frac{\ell_{n+1}}{\log 1/t} \right), \quad \text{as } t \to 0,
\]

and if

\[
B_{n} = O(n^{-\delta}), \quad \delta > 0,
\]

then

\[
\lim_{n \to \infty} \left[ C_{\alpha}(\lambda n) - C_{\alpha}(n) \right] = \frac{2\lambda}{\pi} \log \lambda, \quad \lambda > 1,
\]

where \( C_{\alpha}(n) \) denotes the Cesàro mean of order \( \alpha \) of the conjugate series 
(1.1.2), at \( \theta = x \).

3.1. We shall now prove the analogous theorem for the \((C, \alpha)\) summability of the Fourier series for \(-1 < \alpha < 0\).

**Theorem 4.** If, for \(-K < \alpha < 0\),

\[
\int_{0}^{1} \left| d\Phi_{n+1}(u) \right| = o \left( \frac{\ell_{n+1}}{\log 1/t} \right),
\]

as \( t \to 0 \), and if

\[
A_{n} = O(n^{-\delta}), \quad \delta > 0,
\]

then the Fourier series (1.1.1), at \( \theta = x \), is summable \((C, \alpha)\) to sum \( s \).

The following lemmas will be required for the proof of Theorem 4.

**Lemma 7 [7].** If \( k^{\alpha}(n, t) \) denotes the \((C, \alpha)\) mean of the series

\[
\frac{1}{\pi} + \frac{2}{\pi} \sum_{r=1}^{\infty} \cos \nu t,
\]

\[\text{See [13, Theorem 4].}\]
and if

\[ K^\alpha(n, t) = \frac{2}{\pi} \frac{\sin(n; t)}{A_n^\alpha (2 \sin t/2)^{\alpha+1}}, \]

where

\[ A_n^\alpha = \frac{\Gamma(\alpha + 1 + n)}{\Gamma(\alpha + 1) \Gamma(n + 1)}, \quad (n; t) = \left( n + \frac{1}{2} + \frac{\alpha}{2} \right) t - \frac{\alpha \pi}{2}, \]

then for \(-1 < \alpha < 1,\)

\[ | k^\alpha(n, t) - K^\alpha(n, t) | < A n^{-1} t^{-2}. \]

**Lemma 8.** If, for \(-1 < \alpha < 0,\)

\[ M(n, u) = \frac{1}{\Gamma(-\alpha)} \int_u^x (t - u)^{-\alpha-1} n \Delta k^{\alpha+1}(n, t) dt, \quad r < 1, \]

then

\[ | M(n, u) | \begin{cases} < A n^{\alpha+1}, & u \leq 1/n, \\ < A u^{-\alpha-1}, & u \geq 1/n, \end{cases} \]

the results holding uniformly in \(X.\)

The proof of Lemma 8 is exactly similar to that of Lemma 3.

3.2. **Proof of Theorem 4.** Writing \(\tilde{\Phi}(t) = \frac{1}{2} \{ f(x+t) + f(x-t) \} \) and denoting by \(\tilde{\tau}_{\alpha+1}(n)\) the \((C, \alpha+1)\) mean of the sequence \(\{ n A_n(x) \},\) we have proceeding as in the proof of Theorem 1,

\[ \tilde{\tau}_{\alpha+1}(n) = \int_0^x \tilde{\Phi}(t) n \Delta k^{\alpha+1}(n, t) dt = \int_0^x \Phi(t) n \Delta k^{\alpha+1}(n, t) dt \]

\[ = \int_0^{n-r} + \int_{n-r}^x = J + I, \]

say. Now it is easily seen that

\[ J = \int_0^{n-r} d\tau_{\alpha+1}(u) M(n, u), \]

where

\[ M(n, u) = \frac{1}{\Gamma(-\alpha)} \int_u^{n-r} (t - u)^{-\alpha-1} n \Delta k^{\alpha+1}(n, t) dt. \]

Now taking help from Lemma 8, we obtain \(J = o(1)\) as in the proof of Theorem 1. Also
\[ I = \int_{n-r}^{r} \phi(t) n \Delta k^{a+1}(n, t) dt = (\alpha + 1) \int_{n-r}^{r} \phi(t) \{ k^{a}(n, t) - k^{a+1}(n, t) \} dt \]

\[ = (\alpha + 1) \int_{n-r}^{r} \phi(t) \{ k^{a}(n, t) - K^{a}(n, t) dt \} \]

\[ - (\alpha + 1) \int_{n-r}^{r} \phi(t) \{ k^{a+1}(n, t) - K^{a+1}(n, t) \} dt \]

\[ + (\alpha + 1) \int_{n-r}^{r} \phi(t) \{ K^{a}(n, t) - K^{a+1}(n, t) \} dt = o(1), \]

as before. This shows that under the condition of the theorem the sequence \( \{ nB_{a}(x) \} \rightarrow O(C, \alpha + 1) \). The proof now follows from the fact that the hypotheses imply the summability \((C)\) of (1.1.1), at \( \theta = x \), and therefore also summability \((A)\) and hence by Lemma 7 summability \((C, \alpha)\).

References


University of Gorakhpur, Gorakhpur, India