BISHOP'S GENERALIZED STONE-WEIERSTRASS THEOREM FOR THE STRICT TOPOLOGY

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1. Let X be a locally compact Hausdorff space, $C(X)_\beta$ the locally convex topological vector space obtained from all bounded complex continuous functions on X by employing the strict topology [2]. The present note is devoted to a version of Bishop's generalized Stone-Weierstrass theorem [1] applicable to certain subspaces of $C(X)_\beta$; essentially it is a footnote to an earlier paper [4], in which a modification of de Branges' proof of the Stone-Weierstrass theorem [3] was used to obtain Bishop's theorem. Insofar as possible the notation will be that of [4].

The version of Bishop's theorem we shall write down was motivated by, and has application to, the spectral theory of bounded continuous functions on locally compact abelian groups, where the strict topology enjoys a useful rôle [5]. However, our applications to spectral theory amount at best to new proofs of known results.

Recall that the strict topology has a base of neighborhoods of $0 \in C(X)_\beta$ of the form

$$V = \{f \in C(X)_\beta : \|fg\| \leqq 1\}, \tag{1}$$

where $g \in C_0(X)$, the space of continuous functions vanishing at infinity on X, and the norm is the usual supremum norm [2]. Moreover, the vector space of continuous linear functionals on $C(X)_\beta$ is precisely the vector space $M(X)$ of all finite complex regular Borel measures on X, with the pairing $\langle f, \mu \rangle = \int f d\mu$, $f \in C(X)_\beta$, $\mu \in M(X)$ [2].

For any subsets B of $C(X)_\beta$ and K of X we call K *an antisymmetric set* for B if, for f in B, (the restriction) $f|K$ being real valued implies $f|K$ is constant. If K is a closed subset of X, hence locally compact, then $C(K)_\beta$ is of course well defined, with its own strict topology. Our Bishop theorem for the strict topology is the following.

THEOREM. *Let A be a closed subspace of $C(X)_\beta$, B a subset of $C(X)_\beta$ for which $BA = \{ba : b \in B, a \in A\} \subset A$. Then*

(a) *Every antisymmetric set for B is contained in a maximal antisymmetric set, and the collection $\mathcal{K}_B$ of maximal antisymmetric sets for B forms a closed, pairwise disjoint covering of X.*

(b) *For $f \in C(X)_\beta$, if $f|K$ is in the closure of $A|K$ in $C(K)_\beta$, for*

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each K in $\mathcal{K}_B$, then $f \in A$.

The proof of (a) is exactly the same as that given in [**4**, Theorem 1.1]. Our proof of (b) will be simply a modification of that of [**4**, Theorem 1.1(a)]. We require only the following simple

LEMMA. *Let V be the neighborhood* (1) *of* 0 *in* $C(X)_\beta$, $V^\circ = \{\mu \in M(X): |\int h d\mu| \leqq 1, h \in V\}$. *Then V° is the set of measures μ in $M(X)$ vanishing off $\{x: g(x) \neq 0\}$ for which $\|g^{-1}\mu\| \leqq 1$ (in the usual measure norm).*

(Here $g^{-1}\mu$ denotes the usual product of a function and a measure.)

PROOF. Let $\mu \in V^\circ$, so $|\int f d\mu| \leqq 1$ if $\|fg\| \leqq 1$. Let F be any compact Baire set in $E = X \setminus \{x: g(x) \neq 0\}$, $c > 0$. Since F is a G_δ we clearly can find a sequence $\{f_n\} \subset C(X)_\beta$ which decreases (pointwise) to $c\phi_F$ (ϕ_F the characteristic function), with $\|f_n g\| \leqq 1$. Consequently by monotone convergence $|c\int \phi_F d\mu| \leqq 1$, and, since $c > 0$ is arbitrary, $\mu(F) = 0$. So μ vanishes on all Baire sets contained in E. On the other hand if F is now just a compact subset of E then, by regularity, for $\epsilon > 0$ we have an open Baire set U containing F for which $|\mu|(U \setminus F) < \epsilon$, hence $\epsilon > |\mu|((E \cap U) \setminus F) \geqq |\mu((E \cap U) \setminus F)| = |\mu(E \cap U) - \mu(F)|$, where $|\mu|$ is the usual absolute value (= total variation) measure associated with μ. But $E \cap U$ is a Baire set contained in E, so $\mu(E \cap U) = 0$, and $|\mu(F)| < \epsilon$. So μ vanishes on compact subsets of the closed set E, hence on all Borel sets contained in E.

Consequently $g^{-1}\mu$ makes sense as a σ-finite measure; but

$$\left| \int fg \cdot g^{-1} d\mu \right| \leqq 1 \text{ if } \|fg\| \leqq 1, \qquad f \in C(X)_\beta,$$

and since any continuous h of norm 1 with compact support $\subset \{x: g(x) \neq 0\}$ is of the form fg with $\|fg\| \leqq 1$, we have $|\int h g^{-1} d\mu| \leqq 1$ for all such h. Consequently $|g^{-1}\mu|(K) \leqq 1$ for any compact $K \subset \{x: g(x) \neq 0\}$, and $\|g^{-1}\mu\| \leqq 1$ as was to be shown.

Conversely, if μ vanishes off $\{x: g(x) \neq 0\}$, $\|g^{-1}\mu\| \leqq 1$, and $\|fg\| \leqq 1$, then

$$\left| \int f d\mu \right| = \left| \int f g g^{-1} d\mu \right| \leqq \|fg\| \, \|g^{-1}\mu\| \leqq 1,$$

and the lemma is proved.

Now suppose $f \in C(X)_\beta$ satisfies the hypotheses of (b), while $f \notin A$. Then for some V as in (1) we have $(f + 2V) \cap A = \varnothing$, so $(f + V) \cap (A + V) = \varnothing$. Thus by the Hahn-Banach theorem we have a μ in $M(X)$ with $|\int f d\mu| > 1 \geqq |\langle A + V, \mu \rangle|$, whence $\mu \in A^\perp$ (the measures annihilating A), and $\mu \in V^\circ$. So $\mu \in A^\perp \cap V^\circ$, $\int f d\mu \neq 0$.

By the Krein-Milman theorem (and the fact that $A^{\perp}\cap V^{\circ}$ is weak* compact and convex) we must have an extreme point μ of $A^{\perp}\cap V^{\circ}$ for which $\int f d\mu \neq 0$. Now suppose (as we shall see in a moment) that μ is carried by an element K of $\mathcal{K}_B$. Then μ provides a continuous functional on $C(K)_\beta$, vanishing on $A|K$, and thus at f by the hypothesis of (b), which is the desired contradiction.

So it only remains to show our extreme μ is carried by some K in $\mathcal{K}_B$, hence (in view of (a)) that carrier μ (defined as the complement of open sets of $|\mu|$ measure zero) is an antisymmetric set for B.

Let $K=$carrier μ, and suppose $h'|K$ is real for some $h'\in B$. Since h' is bounded we can certainly choose nonzero real c_1, c_2 so that $0<c_1h'+c_2<1$ on K, and it will suffice to show $h=c_1h'+c_2$ is constant on K. Clearly $hA\subset A$, and so for a measure ν in $A^{\perp}$, $h\nu\in A^{\perp}$, and thus $(1-h)\nu=(\nu-h\nu)\in A^{\perp}$. Now[2] $hg^{-1}\mu$ and $(1-h)g^{-1}\mu$ are nonzero measures so (as in [4]) we may write

$$\mu = \|hg^{-1}\mu\| \frac{h\mu}{\|hg^{-1}\mu\|} + \|(1-h)g^{-1}\mu\| \frac{(1-h)\mu}{\|(1-h)g^{-1}\mu\|},$$

where

$$\begin{aligned}\|hg^{-1}\mu\| + \|(1-h)g^{-1}\mu\| &= \int |hg^{-1}|\, d|\mu| + \int |(1-h)g^{-1}|\, d|\mu| \\ &= \int h|g^{-1}|\, d|\mu| + \int (1-h)|g^{-1}|\, d|\mu| \\ &= \int |g^{-1}|\, d|\mu| = \|g^{-1}\mu\| \leqq 1.\end{aligned}$$

Observing that $h\mu/\|hg^{-1}\mu\|$ and $(1-h)\mu/\|(1-h)g^{-1}\mu\|$ lie in V° by our lemma, hence in $A^{\perp}\cap V^{\circ}$, we thus have $\mu=h\mu/\|hg^{-1}\mu\|$ since μ is extreme; and this of course implies h is constant on $K=$carrier μ, completing the proof of the theorem.[3]

Note that replacing B by its strictly closed span in $C(X)_\beta$, or by the strictly closed subalgebra of $C(X)_\beta$ it generates, yields a finer decomposition $\mathcal{K}$ of X, and thus one might as well take B to be a strictly closed subalgebra of $C(X)_\beta$ to begin with.

As a simple consequence of the theorem we should probably note the following Stone-Weierstrass theorem for $C(X)_\beta$, which strength-

[2] Here g is our function defining V in (1), and μ our extreme element of $A^{\perp}\cap V^{\circ}$.

[3] The same result holds for the space of all continuous functions in the compact-open topology, where the above proof simplifies.

ens a version given by Buck [2] (where it is assumed that A contains a nowhere vanishing function).

COROLLARY. *Let A be a closed self-adjoint subalgebra of $C(X)_\beta$ which separates X in the sense that for any ordered pair x_1, x_2 of distinct points of X there is an f in A with $f(x_1)=1, f(x_2)=0$. Then $A=C(X)_\beta$.*

Here we take $B=A$, and note that any antisymmetric set for A reduces to a single point x; since $A \mid \{x\}$ coincides with the full set of constants by hypothesis, every f in $C(X)_\beta$ is in A by (b).

2. Let G be a locally compact abelian group, $G^\wedge$ the dual (character) group, which we may view as a subset of $C(G)_\beta$. (In fact, since the strict topology and the compact open topology coincide on uniformly bounded sets in $C(G)$, $G^\wedge$ is a (topological) subspace, and, of course, a multiplicative group in $C(G)_\beta$.)

Let H be a closed subgroup of G, $H^\perp$ the orthogonal subgroup in $G^\wedge$ of characters identically 1 on H, and F a closed subset of $G^\wedge$ with $H^\perp \cdot F \subset F$. Then for the (algebraic) subspaces of $C(G)_\beta$ these sets span, we have $(\text{span } H^\perp)(\text{span } F) \subset \text{span } F$, whence (by separate continuity of multiplication in the strict topology) $(\text{span } H^\perp)^- \cdot (\text{span } F)^- \subset (\text{span } F)^-$ holds for their strict closures, and we may apply our result to $B=(\text{span } H^\perp)^-$, $A=(\text{span } F)^-$.

Now we can identify $H^\perp$ with the character group of G/H, and so identify $\mathcal{K}_B$ as the set of cosets mod H; for each coset is a set of constancy of B and thus an antisymmetric set, while for any larger subset E, $h \pm \bar{h} = h \pm h^{-1}$ provides a nonconstant real or purely imaginary function on E for some h in $H^\perp$. So by (b) we have

$$(2)\quad f \in (\text{span } F)^- \text{ if } f \mid (x+h) \in \{(\text{span } f) \mid (x+h)\}^- \text{ in } C(x+H)_\beta$$

for each x in G. With $f_x(y)=f(x+y)$, the hypothesis of (2) amounts precisely to $f_x \mid H \in \{(\text{span } F) \mid H\}^-$ in $C(H)_\beta$ for every x in G; and since restricting an element of $F \subset G^\wedge$ to H corresponds to mapping it canonically into $G^\wedge/H^\perp = H^\wedge$, if F_0 is the image of our F in $G^\wedge/H^\perp$ we have

$$(3)\qquad f \in (\text{span } F)^- \text{ if } f_x \mid H \in (\text{span } F_0)^- \text{ in } C(H)_\beta,$$

for all x in G. Consequently (3) holds if F_0 is a closed subset of $G^\wedge/H^\perp$ and F is its preimage in $G^\wedge$ (a fact which is implicit in [6], and contained in the statement of [5, 5.8]).

From this we can easily obtain (half of) a result of Reiter [6]: if F_0 is a set of spectral synthesis then F is also. For if the bounded continuous function f on G has its spectrum contained in F then of

course the same is true for f_x, and, as is easily proved (as in the first half of [**5**, 5.6]) $f|H$ has its spectrum contained in F_0. Thus spectrum $(f_x|H) \subset F_0$, which implies $f_x|H \in (\text{span } F_0)^-$ since F_0 is a set of spectral synthesis, for every x in G. So by (3) $f \in (\text{span } F)^-$ for every such f, i.e., F is a set of spectral synthesis.

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