exists a g in $B^\wedge|K$ such that $fg|V=1$. It follows that $B^\wedge|K$ is a regular algebra (in the sense of Šilov) and a standard argument (see [4, proof of Lemma 25 E, p. 85]) implies that $B|K^\wedge=C(K)$.[2]

COROLLARY. *$B^\wedge=C(\Omega_0)$ if and only if $\alpha(x)$ has no singular points.*

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[2] We are indebted to Professor I. Glicksberg for supplying us with the argument in this paragraph.

THE MAXIMAL IDEAL SPACE OF THE FUNCTIONS LOCALLY IN A FUNCTION ALGEBRA

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I. **Introduction.** If A is a function algebra[1] with maximal ideal space, M, it is natural to consider those continuous functions on M which belong "locally" to A. Precisely, if f is a complex-valued continuous function on M, we shall say that *f is locally in A* provided that *for each m in M there is a neighborhood, U of m and a function a in A, such that $f|_U=a|_U$.*

The outstanding problem concerning functions locally in A is: *Must every function which is locally in A necessarily be a member of A?* This note makes no significant contribution toward the solution of this problem. Its, very limited, aim is to study the maximal ideal space of the algebra generated by the functions which are locally in A. We prove, making essential use of the Local Maximum Modulus Principle [2], that this maximal ideal space is *the same* as the maximal

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[1] Definitions are given in §II.

ideal space of A. (We remark that the Local Maximum Modulus Principle also shows that the corresponding Šilov boundaries are the same.) Actually, our proof is carried out for a larger class of functions than those which are locally in A. We show, by example, that for certain A, this class includes functions which are *definitely not in* A.

We are indebted to Professor K. Hoffman for conversations from which this work stems.

II. We first recall a few definitions and facts. (See also [1, Chapter IV].)

II.1. Let X be a compact Hausdorff space, and let $\mathcal{C}(X)$ be the Banach algebra of all complex-valued continuous functions on X, normed by $\sup_X|\cdot|$. A *function algebra on* X is a closed subalgebra of $\mathcal{C}(X)$ which contains the constants and separates the points of X.

II.2. If A is a function algebra on X, its *maximal ideal space*, M, may be described as the largest space containing X to which A extends as a function algebra. The algebra homomorphisms of A onto the complex numbers are all given by evaluation at the points of M. Every such homomorphism is continuous and of norm one.

II.3. If f is a complex-valued continuous function on M, we let (A, f) denote the function algebra on M which is generated by A and f.

REMARK. To prove that the maximal ideal space of the function algebra on M, generated by the functions locally in A, is M, it is enough to show that *if f is locally in A, then the maximal ideal space of (A, f) is M*. The following theorem includes this result as a special case.

II.4. THEOREM. *Let A be a function algebra with maximal ideal space, M. Let f be a continuous function on M for which there exists a decomposition of M into a finite number of subsets, $S_1 \cup \cdots \cup S_n = M$, together with functions $a_1, \cdots, a_n$ in A, such that $f|_{S_i} = a_i|_{S_i}$, $i = 1, \cdots, n$. Then the maximal ideal space of (A, f) is M.*

PROOF. Since (A, f) is normed by $\sup_M|\cdot|$, if q is any point in the maximal ideal space of (A, f) we have

$$|e(q)| \leqq \sup_M |e|, \qquad \text{for all } e \text{ in } (A, f). \tag{1}$$

Furthermore, "evaluation at q" induces an algebra homomorphism of A onto the complex numbers. So there must be a point, p in M, such that

$$\mathbf{a}(q) = \mathbf{a}(p), \qquad \text{for all } \mathbf{a} \text{ in } A. \tag{2}$$

Our task is to show that $f(q) = f(p)$. To do so, let us assume the contrary.

$$f(q) \neq f(p). \tag{3}$$

The conditions, $f|_{S_i} = a_i|_{S_i}$ and $S_1 \cup \cdots \cup S_n = M$, lead directly to the basic identity

$$\prod_{i=1}^{n} (f - a_i) = 0; \tag{4}$$

from which, by applying (2), we also get

$$\prod_{i=1}^{n} (f(q) - a_i(p)) = 0. \tag{5}$$

By renumbering, we can arrange that

(6) $f(q) = a_i(p)$ for $i = 1, \cdots, r$ and $f(q) \neq a_j(p)$ for $j = r+1, \cdots, n$ with $0 < r < n$.

(Condition (5) excludes $r = 0$. Conditions (3) and (4) together forbid $r = n$.)

Set

$$g = \prod_{i=1}^{r} (f - a_i) \quad \text{and} \quad h = \prod_{j=r+1}^{n} (f - a_j).$$

Then g and h belong to (A, f). And, by (3) and (6), we have

$$g(p) \neq 0 \quad \text{and} \quad h(q) \neq 0. \tag{7}$$

Let $G = [m \in M: g(m) = 0]$ and $H = [m \in M: h(m) = 0]$. Then G and H are closed subsets of M whose union is all of M. Since $h(q) \neq 0$, we know, by (1), that G is not empty. Observe also

(8) p is in $H - (G \cap H)$, which is an open and closed subset of $M - (G \cap H)$.

Next, define

$$t = \prod_{i=1}^{r} \prod_{j=r+1}^{n} (a_i - a_j).$$

Then t belongs to A. Moreover, from (2), (6) and (7), we see that $t(p) = (h(q))^r \neq 0$. Finally, notice that

$$G \cap H \subset [m \in M: t(m) = 0]. \tag{9}$$

LEMMA. *There is a function b in A, such that $|b(p)| > 1 > \sup_G |b|$.*

PROOF OF LEMMA. Consider the uniform closure on G of the functions in A. Its maximal ideal space is precisely

$$\hat{G} = [m \in M: |a(m)| \leqq \sup_G |a|, \text{ for all } a \in A].$$

Now *suppose* p were in $\hat{G}$. Let $S = [y \in \hat{G}: |t(y)| \geqq |t(p)|]$. Then p is in S. Since $t(p) \neq 0$, condition (9) implies that $S \cap (G \cap H)$ is empty. Let S_p be the connected component of S containing p. By (8), it must be that S_p is contained in $H - (G \cap H)$. However, the Local Maximum Modulus Principle asserts (see [2, p. 9]) that S_p intersects G. Contradiction.

Hence, p is not in $\hat{G}$. So, there must be some function b in A, such that $|b(p)| > 1 > \sup_G |b|$. Thus, the lemma is established.

Since[2] $b(p) = b(q)$ and $h(q) \neq 0$, there is a positive integer N, such that $|h(q) \cdot b^N(q)| > \sup_G |h \cdot b^N|$. Evidently, $h \cdot b^N$ is in (A, f) and attains its maximum modulus over M on G. Hence, $|(h \cdot b^N)(q)| > \sup_M |h \cdot b^N|$, even though $h \cdot b^N$ belongs to (A, f). But this contradicts (1), and we are done.

III. **An example.** It is not difficult to construct an A and an f, as in the theorm in §II.4, for which $(A, f) \neq A$. For example, let B be the bicylinder in complex 2-space defined by $|z| \leqq 1$, $|w| \leqq 1$. Set

$$S_1 = B \cap [z = 0], \qquad S_2 = B \cap [z = w],$$
$$S_3 = B \cap [w = 0], \quad \text{and} \quad M = S_1 \cup S_2 \cup S_3.$$

Take A to be the uniform limits on M, of the polynomials in z and w. It is easily verified that M is the maximal ideal space of A.

Define f to be 0 on S_1, z on S_2, and 0 on S_3. Then f is a continuous function on M which agrees, on each S_i, with an element of A. *If f were in A*, there would be a sequence of polynomials P_j converging uniformly to f on M, with $P_j(0, 0) = 0$ for all j. Separating out terms we can write $P_j(z, w) = A_j(z) + B_j(z, w) + C_j(w)$, where the polynomials $B_j(z, w)$ contain *no first order terms*. On S_1, $P_j(z, w) = C_j(w)$ and converges to 0. On S_2, $P_j(z, w) = A_j(z) + B_j(z, z) + C_j(z)$ and converges to z. On S_3, $P_j(z, w) = A_j(z)$ and converges to 0. It follows that on S_2, $B_j(z, z)$ converges to z. However, since each polynomial $B_j(z, z)$ is divisible by z^2, this must remain true in the (uniform) limit, and we have a contradiction. Hence, f is not in A.

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[2] The referee shortened our original argument here.