In this paper we prove two theorems relating positive definite measures to induced representations. The first shows how the injection of a positive definite measure on a topological group $H$ into a containing locally compact group $G$ in which $H$ is closed gives rise to induced representations. The second is another version of Mackey's imprimitivity theorem, along the lines of Loomis' proof [5]. We feel this is justified on several grounds. Firstly, our proof is simpler than Loomis'. We make no use of the Radon-Nikodym theorem nor of quasi-invariant measures. Secondly, we do not assume in advance that our system of imprimitivity is based on the reduced algebra of Borel sets in $G/H$. Instead, this fact is seen as a consequence of the theorem. Finally, the statement and proof of Theorem 2 in [5] are in need of minor repairs. Using Loomis' notation, the induced representation space of $V$ is spanned, not by the set $\{f_u : u \in H\}$, but rather by the set $\{[E]f_u : u \in H, E \text{ a Borel subset of } G/K\}$. Formula (8) must then be replaced by formula (11) in the statement of the theorem. The algebra $C_0(S\times G)$ used in the present paper may be looked upon as a device for accomplishing these changes.

All nonobvious definitions, notations, and conventions are those of [1].

1. Let $f, g \in C_0(G)$. Define $f \circ g$ and $f^*$ by

$$(f \circ g)(x) = \int f(y)g(xy^{-1})\,dy$$

and

$$f^*(x) = [f(x^{-1})^-]^{-\delta_0(x)^{-1}}.$$ 

$C_0(G)$, equipped with $\circ$, $\ast$, and the usual inductive limit topology, is a topological $\ast$-algebra. This is a group algebra with multiplication defined in a way differing slightly from the usual one. If $x \in G$, we define $(R(x)f)(y) = f(yx)$. The map $(x, f) \mapsto R(x)f$ is continuous.

A measure $\mu$ on $G$ such that $\mu(f^* \circ f) \geq 0$ for all $f \in C_0(G)$ is called positive definite. Given such a $\mu$, one defines a pseudo-Hilbert inner product on $C_0(G)$ by setting $(f, g)_{\mu} = \mu(g^* \circ f)$. One then completes
$C_0(G)$ to get a Hilbert space $\mathcal{H}_\mu$ and, for each $x \in G$, extends $R(x)$ to a unitary operator $R_\mu(x)$ on $\mathcal{H}_\mu$. $R_\mu$ is then a unitary representation of $G$ on $\mathcal{H}_\mu$.

**Theorem 1.** Let $H$ be a closed subgroup of the locally compact group $G$. Let $\mu$ be a positive definite measure on $H$. Let $\nu$ be the measure on $G$ obtained by injecting $\frac{1}{2}\mu$ into $\mathcal{H}_\mu$. Then $\nu$ is positive definite. Moreover, $R_\mu$ is unitarily equivalent to $U^{R_\mu}$ via the closure $V$ of the map $f \mapsto f$ of $C_0(G)$, where, for $x \in G$, $f(x)$ is the vector in $\mathcal{H}_\mu$ defined by $f(x)(\xi) = \delta_{H}(\xi)^{-1/2} \delta_{G}(\xi)^{1/2} f(x)$.

**Proof.** Let $f \in C_0(G)$ and choose $h \geq 0$ in $C_0(G)$ such that $\int_H h(\xi) d\xi = 1$ for all $x \in G$ such that $f(x) \neq 0$. Then

$$
\int (f \ast f)(\xi) \delta_{H}(\xi)^{-1/2} \delta_{G}(\xi)^{1/2} d\mu(\xi)
= \int \int \int h(\eta) f(\xi) \delta_{H}(\xi)^{-1/2} \delta_{G}(\xi)^{1/2} d\eta d\mu(\xi)
= \int \int \int h(x) [f(\eta^{-1} x)] \delta_{H}(\eta)^{-1/2} \delta_{G}(\eta)^{1/2} d\eta d\mu(\xi)
= \int \int \int h(x) [f(\eta x)] \delta_{H}(\eta)^{-1/2} \delta_{G}(\eta)^{1/2} \delta_{H}(\xi)^{-1} \delta_{G}(\xi)^{-1} d\eta d\mu(\xi)
= \int \int \int h(x) [f(x)] \delta_{H}(\xi)^{-1} \delta_{G}(\xi)^{-1} d\eta d\mu(\xi)
= \int \int \int h(x) (f(x)(\eta) - f(x(\eta))) d\eta d\mu(\xi)
= \int h(x)(f(x)(\eta)) d\eta d\mu(\xi)
\geq 0.
$$

Thus $\nu$ is positive definite.

It is trivial to verify that $f \in \mathcal{H}_\theta$ (see [1, §2] for the definition). Therefore $f$ is in the Hilbert space $\mathcal{H}$ of $U^{R_\mu}$. Moreover, the above equations, together with the definition of the norm in $\mathcal{H}$, show that $\|f\| = \|f\|$. Hence the isometry $V$ is well defined. Since $V$ clearly sets up an equivalence between $R_\mu$ and a subrepresentation of $U^{R_\mu}$, we only have left to show that $V$ is onto.

Let $g \in C_0(G)$, $u \in C_0(H)$. Regarding $u$ as a member of $\mathcal{H}_\mu$, we may form

$$
e(g, u)(x) = \int_H \delta_{H}(\xi)^{-1/2} \delta_{G}(\xi)^{1/2} g(\xi) R_\mu(\xi)^{-1} u d\xi$$


as in [1]. Since this integral converges in \( C_0(H) \), we obtain

\[
\epsilon(g, u)(x)(\eta) = \int_H \delta_H(\xi)^{-1/2}\delta_0(\xi)^{1/2}g(\xi x)u(\eta^{-1}d\xi
\]

\[
= \int_H \delta_H(\xi\eta)^{-1/2}\delta_0(\xi\eta)^{1/2}g(\xi\eta x)u(\xi^{-1}d\xi.
\]

It is now easy to see that if we set \( f(x) = \int_H \delta_H(\xi)^{-1/2}\delta_0(\xi)^{1/2}g(\xi x)u(\eta^{-1}d\xi \), then \( f \in C_0(G) \) and \( \dot{f} = \epsilon(g, u) \). Thus \( \{ f : f \in C_0(G) \} \) is dense in \( \mathcal{C} \) by [1, Lemma 2b], and \( V \) is onto.

2. Let \( (S, G) \) be a locally compact transformation group (with \( G \) acting on the right). By a unitary representation of \( (S, G) \) on the Hilbert space \( \mathcal{C} \) we shall mean a \(*\)-representation \( E \) of \( C_0(S) \) (under the pointwise operations) in \( L(\mathcal{C}, \mathcal{C}) \) together with a unitary representation \( U \) of \( G \) on \( \mathcal{C} \) such that:

1. \( E(C_0(S)) \mathcal{C} \) is dense in \( \mathcal{C} \);
2. \( U(x)E(f)U(x^{-1}) = E(R(x)f), \ x \in G \), where \( (R(x)f)(\varphi) = f(px) \).

Note that from the \(*\)-representation property of \( E \) it follows that \( E \) is continuous from \( C_0(S) \) in the \( \| \cdot \|_\infty \) norm to \( \mathcal{L}(\mathcal{C}, \mathcal{C}) \) in the uniform norm.

As an example, let \( G \) be a locally compact group and \( H \) a closed subgroup. Let \( S = G/H \) (right cosets) and let \( G \) operate on \( S \) in the usual way. Let \( \pi \) be the canonical projection of \( G \) onto \( S \). Let \( L \) be a unitary representation of \( H \). Form the induced representation \( U^L \) of \( G \), operating on the Hilbert space of functions \( \mathcal{C} \). For \( f \in C_0(S) \), define \( E^L(f) \) on \( \mathcal{C} \) by setting \( (E^L(f)g)(x) = f(\pi(x))g(x) \). It is easily verified that this definition makes sense and that \( (E^L, U^L) \) is a unitary representation of \( (S, G) \) on \( \mathcal{C} \). It is called the unitary representation of \( (S, G) \) induced by \( L \).

Returning now to a general transformation group, let \( f, g \in C_0(S \times G) \). Define \( f \circ g \) and \( f^* \) by \( (f \circ g)(\varphi, x) = \int f(\varphi, y)g(\varphi y^{-1}, xy^{-1})dy \) and \( f^*(\varphi, x) = \int f(\varphi x^{-1}, x^{-1})^{-1}\delta_0(\varphi x^{-1})^{-1} \). It is easily verified that \( \circ \) and \( * \) turn \( C_0(S \times G) \) into a topological \(*\)-algebra (with respect to the usual inductive limit topology on \( C_0(S \times G) \)). Moreover, if \( x \in G \) we define \( (R(x)f)(\varphi, y) = f(\varphi x, yx) \), and if \( h \in C_0(S) \) we define \( (P(h)f)(\varphi, x) = h(\varphi)f(\varphi, x) \). It is easy to see that \( (x, f) \rightarrow R(x)f \) and \( (h, f) \rightarrow P(h)f \) are continuous maps from \( G \times C_0(S \times G) \) into \( C_0(S \times G) \) and \( C_0(S) \times C_0(S \times G) \) into \( C_0(S \times G) \) respectively (even when \( C_0(S) \) is given the sup topology). The algebra \( C_0(S \times G) \) is due to Dixmier [3] and has been studied extensively by Glimm [4].
Let \((E, U)\) be a unitary representation of \((S, G)\) on \(\mathcal{H}\). For \(f \in C_0(S \times G)\) define \(\Phi(f)\) by \(\Phi(f) = \int E(f(\cdot, x)) U(x^{-1})dx\). It is easily verified that \(\Phi\) is a continuous \(*\)-homomorphism from \(C_0(S \times G)\) into \(\mathfrak{L}(\mathcal{H}, \mathcal{H})\) such that \(\Phi(C_0(S \times G))\mathcal{H}\) is dense in \(\mathcal{H}\) and such that \(\Phi(R(x)f) = U(x)\Phi(f)\) and \(\Phi(P(h)f) = E(h)\Phi(f)\) for all \(x \in G\) and \(h \in C_0(S)\). Let \(\nu \in \mathcal{K}\) and define \(\Lambda\) by \(\Lambda(f) = \langle \Phi(f)\nu, \nu \rangle\). \(\Lambda\) is a Radon measure on \(S \times G\) such that \(\Lambda(f^* \circ f) \geq 0\) for all \(f \in C_0(S \times G)\). Any measure on \(S \times G\) satisfying this positivity condition will be called positive definite.

Let \(\Lambda\) be a positive definite measure on \(S \times G\). Exactly as in the case of positive definite measures on groups, we may define a pseudo-Hilbert inner production on \(C_0(S \times G)\) by setting \(\langle f, g \rangle_\Lambda = \langle \Gamma(f^* \circ g) \rangle\). We may then complete \(C_0(S \times G)\) to get a Hilbert space \(\mathcal{K}_\Lambda\). For \(x \in G\), \(R(x)\) extends to a unitary operator \(R_\Lambda(x)\) on \(\mathcal{K}_\Lambda\); for \(h \in C_0(S)\), \(P(h)\) extends to a bounded operator \(P_\Lambda(h)\) on \(\mathcal{K}_\Lambda\). \((P_\Lambda, R_\Lambda)\) is a unitary representation of \((S, G)\) on \(\mathcal{K}_\Lambda\). If, moreover, \(\Lambda\) arises from a unitary representation \((E, U)\) of \((S, G)\) on \(\mathcal{H}\) and a vector \(v \in \mathcal{H}\), as above, and if \(\mathcal{K}_1\) is the smallest \((E, U)\)-invariant subspace of \(\mathcal{H}\) containing \(v\), then \((P_\Lambda, R_\Lambda)\) is unitarily equivalent to the restriction \((E, U)\big|_{\mathcal{K}_1}\) of \((E, U)\) to \(\mathcal{K}_1\) via the closure of the isometry \(f \rightarrow \Phi(f)v\).

Suppose now that \(H\) is a closed subgroup of \(G\) and that \(S = G/H\). For \(h \in C_0(G)\), set
\[
(\tau h)(\pi(x)) = \int_H h(\xi x)d\xi.
\]
For \(k \in C_0(G \times G \times G)\), set \((\sigma k)(\pi(x), y, z) = \int_H k(\xi x, y, z)d\xi\) and \((\theta k)(x, y, z) = k(xz, xy^{-1}, x^{-1})\). Then \(\tau, \sigma, \theta\) are open homomorphisms of \(C_0(G)\) and \(C_0(G \times G \times G)\) onto \(C_0(S)\) and \(C_0(S \times G \times G)\) respectively, and \(\theta\) is a topological automorphism of \(C_0(G \times G \times G)\).

**Lemma.** Let \(\Lambda\) be a measure on \(S \times G\). Define the measure \(\mathcal{M}\) on \(G \times G \times G\) by setting
\[
\int \int \int k(x, y, z)d\mathcal{M}(x, y, z) = \int \int \int (\sigma \theta k)(p, y, z)d\Lambda(p,\gamma)dz
\]
for all \(k \in C_0(G \times G \times G)\). Then there is a measure \(\mu\) on \(G \times G\) such that \(d\mathcal{M}(x, y, z) = dxd\mu(y, z)\). Moreover, for \(\xi \in H\),
\[
d\mu(y\xi, z\xi) = \delta_{\sigma(\xi)}(\delta_{\mu_\xi}(\xi)^{-1})d\mu(y, z).
\]
**Proof.** That \(\mathcal{M}\) factors as above follows from the fact that \(\mathcal{M}\) is invariant under right translation in its first variable (cf. the argument in \([2, \text{bottom of p. 127}]\)). Now let \(k \in C_0(G \times G \times G)\) and \(\xi \in H\). Set \(k_\xi(x, y, z) = k(x, y\xi^{-1}, z\xi^{-1})\) and \(k_\xi(x, y, z) = k(\xi^{-1}x, y, z)\). Then
\[(\sigma \theta k^\ell)(\pi(x), y, z) = \int_H k(\eta x z, y x^{-1} \eta^{-1} x^{-1}, x^{-1} \eta^{-1} x^{-1}) d\eta\]

\[= \delta_H(\xi)^{-1} \int_H k(\xi^{-1} \eta x, y x^{-1} \eta^{-1}, x^{-1} \eta^{-1}) d\eta\]

\[= \delta_H(\xi)^{-1} (\sigma \theta k^\ell)(\pi(x), y, z).\]

From this we obtain

\[\int \int \int k(x, y, z) d\xi d\eta d\xi^* d\eta^* = \int \int \int k^\ell(x, y, z) d\xi d\eta d\xi^* d\eta^*\]

\[= \delta_H(\xi)^{-1} \int \int \int k^\ell(x, y, z) d\xi d\eta d\xi^* d\eta^*\]

and our lemma is proved.

**Theorem 2.** Let \(\Lambda\) be a positive definite measure on \(S \times G\) and define \(\mu\) as in the lemma. For \(\phi, \psi \in C_0(G)\), set \((\phi, \psi)_\mu = \int \phi(y) [\psi(z)]^* d\mu(y, z)\). If \(\xi \in H\), set \((L(\xi)\phi)(y) = \delta_0(\xi)^{1/2} \delta_H(\xi)^{-1/2} \phi(y\xi)\). Then \((\cdot, \cdot)_\mu\) is a pseudo-Hilbert inner product on \(C_0(G)\). Complete \(C_0(G)\) to get the Hilbert space \(\mathcal{V}_\mu\). Then \(L\) extends to a unitary representation \(L_\mu\) of \(H\) on \(\mathcal{V}_\mu\). Finally \((P_\Lambda, R_\Lambda)\) is unitarily equivalent to \((E^{L_*}, U^{L_*})\) via the closure \(W\) of map \(f \rightarrow \hat{f}\) of \(C_0(S \times G)\), where, for \(x \in G\), \(\hat{f}(x)\) is the vector in \(\mathcal{V}_\mu\) defined by \(\hat{f}(x)(y) = f(\pi(x), yx)\).

**Proof.** Let \(f \in C_0(S \times G)\) and \(h \in C_0(G)\). Set \(k(x, y, z) = \overline{h(x)} [f(\pi(x), zx)]^{-1} f(\pi(x), yx)\). Then

\[(\sigma \theta k^\ell)(\pi(x), y, z) = \int_H h(x) f(\pi(x), z) [-f(\pi(xz), yz)] d\xi\]

\[= (\tau h)(\pi(xz)) [f(\pi(xz), z)]^{-1} f(\pi(xz), yz)\]

Hence

\[\int h(x)(\hat{f}(x), \hat{f}(x)) d\mu = \int \int \int (\tau h)(p z) [f(p z, z)]^{-1} f(p z, y z) dz d\Lambda(p, y)\]

\[= \int \int \int f^*(p, z) (\tau h)(p z^{-1}) [f(p z^{-1}, y z^{-1})] dz d\Lambda(p, y)\]

\[= \Lambda(f^* \circ P(\tau h)f) = \langle P_\Lambda(\tau h)f, f \rangle_\Lambda.\]
Now $h \geq 0$ implies that $\tau h \geq 0$, so that $P_{\Delta}(\tau h)$ is a positive operator. Moreover, $x \rightarrow \int f(x) [f(x)]^*$ is continuous from $x$ to $C_0(G)$ so that $x \rightarrow (f(x), f(x))_\mu$ is a continuous function. We conclude that $(f(x), f(x))_\mu \geq 0$ for all $x \in G$. Since $f \rightarrow \int f(x)$ maps $C_0(S \times G)$ onto $C_0(G)$, our first assertion is proved. That $L(\xi)$ is unitary for all $\xi \in H$ follows from the lemma, and that $L_\mu$ is a unitary representation of $H$ is then clear.

Now it is easy to see that $f \in \mathcal{F}_\mu$. Choose $h \in C_0(G)$ so that $\tau h = 1$ on $\{p \in S : f(p, y) \neq 0 \text{ for some } y \in G\}$. Then $P_{\Delta}(\tau h)f = f$ and we obtain $\|f\|_{\Delta} = \|f\|$. Once again we are reduced to showing that $W$ is onto. This is done exactly as in Theorem 1. Let $g, u \in C_0(G)$. Regarding $u$ as a member of $\mathcal{C}_\mu$, form $\epsilon(g, u)$. We obtain $\epsilon(g, u)(x)(y) = \int_H g(\xi)xu(y\xi^{-1})d\xi$. Set $f(\pi(x), y) = \int_H g(\xi)xu(yx^{-1}\xi^{-1})d\xi$. If the supports of $g$ and $u$ are $K_1$ and $K_2$ respectively, the support of $f$ is contained in $\pi(K_1) \times (K_2K_1)$, compact. Hence $f \in C_0(S \times G)$. It is easy to see that $f = \epsilon(g, u)$, and our proof finishes as before.

**Corollary.** Every unitary representation of $(G/H, G)$ is induced.

**Proof.** If the representation space is jointly cyclic under $E$ and $U'$ the corollary follows from the theorem together with the remarks two paragraphs before the lemma. The general case follows from the fact that induction commutes with direct summation.

**References**


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