INTERVAL CLANS WITH NONDEGENERATE KERNEL

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Introduction. The object of this paper is to characterize the clans (compact connected Hausdorff topological semigroups with an identity element) which are homeomorphic to a unit interval and which have a nondegenerate kernel (minimal two-sided ideal). The corresponding case when the kernel is degenerate has been characterized in a paper by H. Cohen and L. I. Wade [2] together with an earlier paper by Mostert and Shields [5].

In a topological semigroup $T$, $K(T)$ or $K$ denotes the kernel of $T$. The symbol $u$ is reserved to denote an identity element. The term “standard thread” will mean a clan with zero which is homeomorphic to a unit interval and whose endpoints are its zero and identity element. In a standard thread $T$ with identity element $u$ and zero $0$, for $a, b \in T$, $[a, b]$ will denote the interval from $a$ to $b$, (or $b$ to $a$) inclusive and $a \leq b$ will mean $a \in [0, b]$, with $a < b$ in case $a \neq b$. A relation $R$ on a topological semigroup $x$ is called a “closed right congruence” if (i) $R$ is an equivalence relation, (ii) $a, b, c \in X$, $aRb$ implies $acRbc$, (iii) $aRx_n$ for $n = 1, 2, \ldots$ and $x_n \rightarrow x$ implies $aRx$ (closed). We denote by $R_a$ the set $\{x \mid aRx\}$. The analog of Theorem I where $R$ satisfies $(2)'$ $(a, b, c \in X, aRb$ implies $caRcb$ (left congruence) instead of $(2)$ is also true.

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**Theorem I.** Let $T$ be a standard thread and $R_a$ closed right congruence on $T$. Then for $a \in T$ either (1) $R_a = a$ or (2) $R_a$ is an interval $[e, b]$ where $e$ is idempotent and $[e, b]$ is a subsemigroup of $T$ with zero element $e$.

**Proof.** Suppose there exists $a' \in T$ such that $a'Ra$ and $a' \neq a$. Let $e = \inf \{x : xRa\}$ and $b = \sup \{x : xRa\}$. Since $R_a$ is closed, $eRa$ and $bRa$. Now $e < b$ which implies $[3]$ that $e = br$ for some $r \geq e$. Therefore, $bRe$ implies $brRer$ implies $eRer$ implies $eRer^*$ for $n = 1, 2, \ldots$ From $[3]$ we know $r^n \rightarrow j = j^2 \leq r$ and hence $eRej$. We will show $j \leq e$. If $j \geq b$, then $bj = b$ and hence $b = bj = (bj)r = br = e$. Therefore $j < b$.
and \( j = bj \leq br = e \), as was to be shown. Now \( j \leq e \) implies \( ej = j \) and since \( eRej, eRj \) which implies by the minimality of \( e \), that \( e = j \) and hence \( e^2 = e \). Now for \( c \in [e, b] \), \( c = bp \) for some \( p \geq c \) and since \( eRb, epRbp \) and we have \( eRc \). This shows that \( R_a = [e, b] \) which is indeed a semigroup with zero element \( e \).

Notice that if \( X \) and \( Y \) are topological semigroups and \( f \) maps \( X \) into \( Y \) continuously and either \( f(xy) = f(x)f(y) \) or \( f(xy) = f(x) \cdot y \), (in case \( Y \) is contained in \( X \)), then the relation \( R \) induced by letting \( R_a = f^{-1}(a) \) is a closed right congruence on \( X \).

**Corollary II.** Let \( X \) be a topological semigroup and let \( A \) be a standard thread contained in \( X \). Then for any element \( c \in X \), \( cA \) is a continuous monotone image of \( A \).

**Proof.** Define the relation \( R \) on \( A \) by \( R_a = \{ x \mid x \in A, \alpha = ca \} \). Then \( R \) is a closed right congruence and hence multiplication by \( c \) is monotone.

**Clans on an interval.** Let \( S \) be a clan which is homeomorphic to a unit interval and which has a nondegenerate kernel, \( K \). By a result of A. D. Wallace [7], the identity element \( u \) of \( S \) is one of the endpoints of \( S \). Note that \( K \) is a closed interval of \( S \) and let (i) \( A = \) the closure of the component of \( S \setminus K \) which contains \( u \), (ii) \( B = S \setminus (K^0 \cup A) \), (iii) \( z = A \cap K \), (iv) \( z' = B \cap K \), and (v) \( d = \) the nonidentity endpoint of \( S \). A result of Faucett [4] is that \( A \) is an abelian subclan of \( S \) with zero \( z \) and that \( K \) consists of either all left zeroes or all right zeroes of \( S \). Let us assume that \( K \) is all right zeroes.

![Figure I](image)

**Lemma III.** \( AB = Ad = B \).

**Proof.** Since \( A \) is a standard thread contained in \( S \), by the analog of Corollary II, \( Ad \) is a continuous monotone image of \( A \) and \( Ad = [zd, ud] = [zd, d] \) which contains \( B \) since \( zd \) is in \( K \). In particular, there exists an element \( a \) in \( A \) such that \( z' = ad \). Now since \( z \) is the zero for \( A \), \( z = za \) and we have \( zd = (za)d = z(ad) = zz' = z' \). Therefore \( Ad = [d, z'] = B \). Also \( AB = A(Ad) = A^2d = Ad = B \) as was to be shown.

Notice that from Corollary II we have \( dA = [dz, du] = [z, d] \) which contains \( z' \). Let \( \theta = \inf \{ a : a \in A, da = z' \} \). Denote \( [\theta, u] \) by \( A_1 \) and \( [z, \theta] \) by \( A_2 \). (Note that \( \theta \neq z \), else \( z' = d\theta = z \).)

**Lemma IV.** \( BA_1 = dA_1 = B \); \( BA_2 = dA_2 = K \); and \( \theta^2 = \theta \).
Proof. In the same manner as the proof of Lemma III we have $dA_1 = [z', d] = B$ and $dA_2 = [z', z] = K$. Since $A_2 \subseteq A_2[3]$, $\theta \in A_2$. Therefore $d\theta^2 \subseteq dA_2 = K$. But $d\theta^2 = z'\theta \in BA_1 = AdA_1 = AB = B$. So $d\theta^2 \subseteq K \cap B = z'$ and by the minimality of $\theta$, $\theta^2 = \theta$. Now we employ a result of Mostert and Shields [5] that $A_1$ and $A_2$ are subclans of $A$, that $\theta$ is a zero for $A_1$ and an identity for $A_2$, and that $a \in A_1$, $a' \in A_2$ implies $aa' = a'a = a'$. Therefore $A_1A_2 = A_2$ and $BA_2 = dA_1A_2 = K$ and the lemma is proved.

Lemma V. $B$ is an abelian subsemigroup of $S$.

Proof. First we show $d^2$ is in $B$. Clearly $d^2$ is not in $A$, else $z' = dz' = d(dB) = d^2B \subseteq A$. So suppose $d^2$ is in $K$. Then $dd^2 = d^2$ and by Lemma IV $d^2 = da$ for some $a$ in $A_2$. Therefore $d^2u = d^2 = dd^2 = d(da) = d'a$, and since left multiplication by $d^2$ induces a closed right congruence on $[u, a]$, $d^2[u, a] = d^2$. Now $a \in A_2$ so that $\theta$ is in $[u, a]$. Hence $d^2\theta = d^2$. But $d^2\theta = d(d\theta) = d(z') = z'$ and $d^2$ is in $B$. Using Lemmas III and IV, we have $B^2 = (dA_1)(dA_2) \subseteq dB A_1 = d(dA_1) \subseteq BA_1 = B$, i.e., $B^2 \subseteq B$. Now we show $B$ is abelian. Using again Lemmas III and IV, $a \in A_1$ implies $da \in B$ and $da = a'd$ for some $a' \in A$. Since $d^2 \subseteq B$, $d^2 = a''d$ for some $a'' \in A$. Using the commutativity of $A$, we have $dad = (a'd)d = a'a''d = a''a'd = a''da = d'a$. Let $b, b'$ be elements of $B$. Then for some $a_1, a_2 \in A_1$, $b = da_1$, and $b' = da_2$. So $bb' = da_1da_2 = d^2a_1a_2 = d^2a_1a_2 = d^2a_1a_1 = b'b$ and $B$ is abelian.

Lemma VI. $KB = z'$.

Proof. Since $KB = dA_2dA_1 = d(A_2d)A_1 \subseteq dBA_1 = dB \subseteq B$, $KB \subseteq K$ and $B = z'$.

In what follows $S/K$ denotes the Rees quotient [6] of $S$ modulo $K$ and $F$ denotes the natural map of $S$ onto $S/K$. Since $d^2 \subseteq B$, $F(d^2) \subseteq F(B)$ and we have

Theorem VII. Let $S$ be an interval clan with a nondegenerate kernel $K$. Let $u$ be the identity and $d$ the nonidentity endpoint of $S/K$; denote $F(K)$ by $0$. Then (i) there exists an element $\theta = \theta^2 \in [0, u] - \{0\}$ such that $d\theta = 0$ and (ii) $d^2 \in [d, 0]$. Further, the function $h: F(A_2) \to K$ defined by $h(x) = d \cdot F^{-1}(x)$ for $x \neq 0$ and $h(0) = z$ is continuous and induces a closed right congruence on $F(A_2)$.

Let $S$ be a clan on an interval $[d, u]$ where $u$ is the identity element. Suppose $S$ has a zero $0$, that (i) and (ii) of Theorem VII are satisfied, that $d \neq 0$ and that $u \neq \theta$. Consider the real interval $[1, 5]$ and define

1. $f: [d, 0] \to [1, 2]$ so that $f(d) = 1, f(0) = 2$ and $f$ is a homeomorphism,
2. \( g: [0, u] \to [3, 5] \) so that \( g(0) = 3, g(\theta) = 4, g(u) = 5 \) and \( g \) is a homeomorphism,
3. \( h: [0, \theta] \to [2, 3] \) so that \( h(0) = 3, h(\theta) = 2, h \) is continuous and \( h \)
induces a closed right congruence \( R \) on \([0, \theta]\), \((R_x = h^{-1}h(x))\).

Further, define for all \( x, y \) and \( z \) on which the functions are defined,
4. \( c \cdot h(x) = h(x), \) all \( c \in [1, 5], \)
5. \( h(x) \cdot g(y) = h(xy), \)
6. \( h(x) \cdot f(y) = 2, \)
7. \( g(x) \cdot g(y) = g(xy), \)
8. \( f(x) \cdot f(y) = f(xy), \)
9. \( g(x) \cdot f(y) = f(xy), \)
10. \( f(x) \cdot g(y) = \{ h(y) \text{ for } y \in [0, \theta], f(xy) \text{ for } y \in [\theta, u] \}. \)

We now show that definition 5 is well defined. The others are clear.
Suppose \( h(a) = h(b) \) and \( a < b \). Then from condition 3 and Theorem I
we have \( a, b \in [e, r] = h^{-1}h(x) \) and \( e^2 = e \) and \( e \) is a zero for \([e, r], \) a
semigroup. Then for \( c \in [0, u], h(a) \cdot g(c) = h(ac) \) and \( h(b) \cdot g(c) = h(bc). \)
If \( c \leq e, \) then \( ac = bc = c \) \([5]\) so that \( h(ac) = h(bc). \) If \( c > e, \) then \( e \leq ac \leq a \)
and \( e \leq bc \leq b \) and \( h(ac) = h(bc) = h(a), \) and \( h \) is well defined.

It can be easily verified that the interval \([1, 5]\) together with defi-
nitions 4 through 10 is a clan \( S' \) with kernel \([2, 3]\) and that \( S'/K \) is
topologically isomorphic to \( S. \)

Now suppose \( S \) in the previous construction were the Rees quotient
of an interval clan \( T = [d, z', z, u] \) with nondegenerate kernel \( K = [z', z]. \)
Then outside \( K, T \) is reproduced by our construction and if \( h \) in 3 is
chosen to be the \( h \) of Theorem VII, the resulting clan is topologically
isomorphic to \( T. \)

If \( d = 0, \) omit definitions 1, 6, 8, 9, and 10; if also \( u = \theta, \) omit the
equation \( g(u) = 5 \) from definition 2; change definition 4 appropriately.
The conclusion is completely analogous for \( S = [2, 4] \) if \( u = \theta \) or
\( S = [2, 5] \) if \( u \neq \theta. \)

From the preceding two paragraphs we conclude:

Theorem VIII. An interval clan with a nondegenerate kernel is char-
acterized by a pair \((S, h)\) where
1. \( S \) is an interval clan with zero \( 0, \) say \( S = [d, 0, u] \) where \( d \) may
equal \( 0, \)
2. \( d^2 \in [d, 0], \)
3. there exists \( \theta = \theta^2 \in (0, u) \) such that \( d\theta = 0, \)
4. \( h \) maps \([0, \theta]\) onto \([1, 2]\) continuously,
5. \( h \) induces a closed right (left in case \( K \) is all left zeroes) congruence
on \([0, \theta]. \)

The pair \((S, h)\) characterizes a particular interval clan \( T \) with a
nondegenerate kernel $K$ in the following sense; $T/K = S$ and the function $h$ of Theorem VII satisfy the conditions of Theorem VIII, and a clan $T'$ gives rise to the same $S$ and $h$ if and only if $T'$ is topologically isomorphic to $T$.

Bibliography


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