ON REGULAR LOCAL RINGS

K. R. MOUNT

This paper generalizes slightly a result of Kunz [1] and Nakai [2]. If $R > S$ are commutative rings with identity we introduce a module $D^*(R/S)$ defined as the quotient of the module $D(R/S)$ of $S$ differentials of $R$ by the submodule consisting of elements which are mapped to zero by every homomorphism of $D(R/S)$ having values in a finitely generated $R$ module. The characteristic exponent of a field is defined to be 1 if the field is of characteristic zero and to be $p$ if the characteristic of the field is $p$. The result is then: If $R$ is a local ring containing a field $k$ of characteristic exponent $p$ such that $D^*(R/k^p)$ is finitely generated, then the following conditions are equivalent: (i) $R$ is a regular local ring. (ii) $D^*(R/k^p)$ is free and if $x$ is an element of the completion of $R$ such that $x^p = 0$ then $x = 0$. (iii) $D^*(R/k^p)$ is free and if $x$ is an element of the form ring of $R$ such that $x^p = 0$ then $x = 0$. We remark that in characteristic zero regularity (under the finiteness condition) is equivalent to the freedom of $D^*(R/k)$ and in any case if the local ring is of the form $A_q$ where $A$ is a finitely generated integral domain and $q$ is a prime the second part of (ii) is automatically satisfied. (See Zariski and Samuel [4, p. 314].)

Lemma 1. If $R > S$ are commutative rings with identity then there is one and only one module $D^*(R/S)$ (to within $R$-isomorphism) satisfying the conditions: (i) There is an $S$-derivation $d^*$ from $R$ to $D^*(R/S)$ such that the image of $d^*$ generates $D^*(R/S)$. (ii) If $h$ is an $S$ derivation from $R$ to a finitely generated $R$ module $M$ then there is an $R$ homomorphism $D^*(h)$ from $D^*(R/S)$ to $M$ such that $D^*(h)d^* = h$. (iii) If $f$ is an element of $D^*(R/S)$ then $h(f) = 0$ for every homomorphism $h$ of $D^*$ to a finitely generated $R$ module if and only if $f = 0$.

Proof. If $F(R/S)$ denotes the collection of elements of $D(R/S)$ annihilated by all $R$ homomorphisms to finitely generated $R$ modules, let $q$ denote the quotient map from $D(R/S)$ to $D(R/S)/F(R/S) = D^*(R/S)$ and set $d^* = qd$ where $d$ is the derivation from $R$ to $D(R/S)$. If $h$ is an $S$ derivation from $R$ to a module $N$ denote by $D(h)$ the homomorphism from $D(R/S)$ satisfying $D(h)d = h$ and suppose $M$ is a second module with properties (i)−(iii) where $d^#$ denotes the derivation from $R$ to $M$ and $D^#(h)$ denotes the homomorphism assigned to a derivation from $R$ to $N$. If $b$ maps $M$ to a finitely generated $R$ module $M$ then $h(f) = 0$ for every homomorphism $h$ of $D^*$ to a finitely generated $R$ module if and only if $f = 0$.

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generated module then one checks easily that \( D(b \, d\#) = bD(d\#) \) and hence if \( f \) is an element of \( F(R/S) \) we have \( D(d\#)(f) = 0 \). Denote by \( D^*(d\#) \) the homomorphism from \( D^*(R/S) \) to \( M \) satisfying the equation \( D^*(d\#)q(f) = D(d\#)(f) \) for \( f \) in \( R \). If \( x \) is an element of \( D(R/S) \) with \( D^*(d\#)q(x) = 0 \) then for a homomorphism \( g \) from \( D(R/S) \) to a finitely generated module \( P \) it follows that \( D^*(g \, d) \) \( D(d\#) = g \), thus \( g(x) = 0 \) and \( D^*(d\#) \) is an isomorphism.

**Lemma 2.** If \( R \) is a local ring containing a field \( K \) such that \( D^*(R/K) \) is finitely generated, or if \( M = D(R/K)/\bigcap m^nD(R/K) \) is finitely generated then \( M = D^*(R/K) \) (\( m \) the maximal ideal of \( R \)).

**Proof.** First \( D^*(R/K) \) is clearly Hausdorff. Denote by \( h \) the quotient map from \( R \) to \( R/m = L \) and note that \( D^*(L/K) = D(L/K) \), using Lemma 1. Let \( A \) be the submodule of \( R/m \otimes D^*(R/K) \) generated by the elements of the form \( 1 \otimes d^*x \) for \( x \) in \( m \) and set \( D^*(L/K) = [R/m \otimes D^*(R/K)]/A \). Define a derivation \( d\# \) from \( R/m \) to \( D^*(L/K) \) by \( d\#(x) = Cl(1 \otimes d^*x) \) where \( Cl(y) \) denotes the coset determined by the element \( y \). If \( f \) is a derivation (over \( K \)) from \( R/m \) to \( M \), a finitely generated \( L \) module, then \( fh \) is a derivation of \( R \) to \( M \) and the map \( 1 \otimes D^*(fh) \) induces a homomorphism \( D^*(f) \) from \( D^*(L/K) \) to \( M \) such that \( D^*(f)d\#(x) = f(x) \). Since \( D^*(L/K) \) is finitely generated it satisfies (i)–(iii) of Lemma 1 and thus \( D^*(L/K) = D^*(L/K) = D(L/K) \).

Denote by \( R(n) \) the ring \( R/m^n \) and by \( m(n) \) the maximal ideal of \( R(n) \) and note that we have the exact sequence \( m(n)/m(n)^2 \rightarrow R(n)/m(n) \otimes D(R(n)/K) \rightarrow D(L/K) \rightarrow 0 \) (Nakai [2, Proposition 9]). It follows easily that \( D(R(n)/K) \) is finitely generated. Now consider an element \( z \) of \( F(R/K) \) and note that if \( h(n) \) denotes the quotient map from \( R \) to \( R(n) \) and if \( d(n) \) is the \( K \) derivation from \( R(n) \) to \( D(R(n)/K) \) then \( D(d(n)h(n)) \) is a homomorphism from \( D(R/K) \) to \( D(R(n)/K) \) with kernel contained in \( m^nD(R/K) \) (Nakai [2, Proposition 9]) and thus \( z \) is an element of \( \bigcap m^nD(R/K) \). If \( M \) is finitely generated then one checks easily that it satisfies conditions (i)–(iii) of Lemma 1.

If \( R \) is a local ring with maximal ideal \( m \) and \( M \) is a Hausdorff \( R \) module in the \( m \)-adic topology denote by \( Co(M) \) the completion of \( M \).

**Lemma 3.** If \( R \) is a local ring containing a field \( K \) with \( D^*(R/K) \) finitely generated then the completion of \( D^*(R/K) \) is isomorphic to \( D^*(Co(R)/K) \).

**Proof.** It will suffice to show that there is an isomorphism from the module \( D^*(R/K)/m^nD^*(R/K) \) to \( D^*(Co(R)/K)/m^nD^*(Co(R)/K) \) which commutes with the quotient maps. Denote by \( D(n)^* \) the
module $D^*(R/K)/m^*D^*(R/K)$ and by $C(n)^*$ the module $D^*(R/K)/m^*D^*(R/K)$. Let $p(n+1/n)$ and $q(n+1/n)$ represent the maps from $D(n+1)^*$ to $D(n)^*$ and from $C(n+1)^*$ to $C(n)^*$ respectively and denote by $p(n)$ and $q(n)$ the quotient maps from $D^*(R/K)$ to $D(n)^*$ and from $D^*(R/K)$ to $C(n)^*$. If $c^*$ denotes the derivation from $Co(R)$ to $D^*(R/K)$ then the derivation $d(n)# = q(n)c^*$ gives rise to a homomorphism $D(d(n)#)$ from $D(R/K)$ to $C(n)^*$ such that $D(d(n)#)d = d(n)#$. If $f$ is an element of $m^*D(R/K)$ then $D(d(n)#)(f) = 0$ and thus there is a homomorphism $J(n)$ from $D(n)^*$ to $C(n)^*$ satisfying the equation $(J(n)p(n))(r(g)) = D(d(n)#)(g)$ for $g$ in $D(R/K)$ and $r$ the quotient map from $D(R/K)$ to $D^*(R/K)$. Suppose $h$ is in $Co(R)$, write $h$ in the form $h = h' + h''$ with $h'$ in $R$ and $h''$ in $m^{n+1}Co(R)$ and set $e(n)[h] = q(n)d*(h')$. The map $e(n)$ defines a $K$ derivation from $Co(R)$ to $D(n)^*$ where we consider $D(n)^*$ as a $Co(R)$ module by setting $h - u = h'$ for $h'$ as above. Denote by $C(v)$ the homomorphism from $D(Co(R)/K)$ to $P$ determined if $v$ is a $K$ derivation from $Co(R)$ to $P$. If $u$ is an element of $m^*D(Co(R)/K)$ then $C(v)(u)$ is an element of $m^*D(R/K)$ whence there is a homomorphism $H(n)$ from $C(n)^*$ to $D(n)^*$ with $H(n)(q(n)s(f)) = C(v)(f)$ for $f$ the quotient map from $D(Co(R)/K)$ to $D^*(Co(R)/K)$. For $x$ in $R$ we have $(H(n)f(n))(p(n)d*x) = p(n)d*x$ and hence $J(n)$ is a monomorphism. If $x$ is in $Co(R)$ then writing $x = x' + x''$ with $x''$ in $m^{n+1}Co(R)$ and $x'$ in $R$ we have that $c^*(x) = c^*(x')$ modulo $m^*Co(R)$ from which it follows that $J(n)$ is onto. To complete the assertion we need only show that $J(n)p(n+1/n) = q(n+1/n)J(n+1)$ and it suffices to show this for the elements of the form $p(n+1/n)d*x$ which one checks easily.

As a consequence we have that if $R = K[[X_1, \ldots, X_n]]$ with $[K; K^p] < \infty$ then $D^*(R/K^p)$ is free on the basis $d^*X_i$ and $d^*Y_j$ where $Y_j$ is a $p$ basis of $K$ over $K^p$. This follows by completing $K[X_1, \ldots, X_n]_K$ where $X$ is the ideal generated by the $X_i$. Also note that if $R$ is any local ring containing a field $K$ such that $D^*(R/K)$ is finitely generated and if $M$ is a hausdorff $R$ module then any homomorphism from $D(R/K)$ to $M$ annihilates $F(R/K)$.

**Proposition 1.** Let $R$ be a local ring containing a field $K$ such that $D^*(R/K)$ is finitely generated. If $R' > R$ with $R'$ regularly quasi-finite over $R$ then $D^*(R'/K)$ is finitely generated.

(For definitions see [3]).

**Proof.** Suppose $R' = R[x_1, \ldots, x_n]$ and assume $R' = R''$ where $m''$ is a maximal ideal of $R''$. Denote by $N''$ the image of $R''$ under the map $d \cdot k$ where $k$ is the inclusion of $R''$ into $R'$ and let $g$ denote
the induced map from $D(R''/K)$ to $D^*(R'/K)$. The image of $g$ is
spanned by the restriction of $g$ to the set $(d \cdot k)(R)$ and by the $g \, dx_i$.
We note first that $D^*(R'/K)$ is a hausdorff $R'$ module and thus is a
hausdorff $R$ module. The map $g(d \cdot k)$ restricted to $R$ is thus a $K$
derivation of $R$ to a hausdorff $R$ module and hence the image of $g$ is
generated by a homomorphic image of $D^*(R/K)$ and by the $g \, dx_i$;
whence $N''$ is finitely generated. Now suppose $f$ is in $R'$. There is an
element $n$ of $R''$, $n$ not in $m''$, such that $nf$ lies in $R''$, thus $d^*(nf) = d^*(n) \cdot f + nd^*(f)$, hence $d^*(f) = (1/n) \cdot h$ where $h$ is in the image of $g$ so $D^*(R'/K)$ is finitely generated.

**Lemma 4.** Suppose $f$ is an epimorphism of the local ring $R$ to the local
ring $R'$ such that $R$ contains a field $K$ with $D^*(R/K)$ and $D^*(R'/K)$
finely generated. If $A = \ker (f)$ then we have the exact sequence
$(R/A) \otimes A \rightarrow R/A \otimes D^*(R/K) \rightarrow D^*(R'/K) \rightarrow 0$.

**Proof.** Denote by $B$ the submodule of $R/A \otimes D^*(R/K)$ generated
by the elements of the form $1 \otimes d^*a$ where $a$ is in $A$ and set
$M = [R/A \otimes D^*(R/K)]/B$. If $h$ is the quotient map of $R/A \otimes D^*(R/K)$
onto $M$ we set $g(x) = h(1 \otimes d^*x')$ where $f(x') = x$ and note that this
defines a map of $R/A$ into $M$ which is independent of the representa-
tion $x'$ and is a $K$ derivation of $R/A$. The induced homomorphism
$H^* = D^*(g)$ from $D^*(R'/K)$ to $M$ is such that $H^*(d^*x) = g(x)$ for $x$ in
$R/A$. On the other hand the module $D^*(R'/K)$ is finitely generated
as an $R$ module and the map $d'^*f$ from $R$ to $D^*(R'/K)$ is a $K$
derivation of $R$, thus $D^*(d'^*f)$ maps $D^*(R/K)$ to $D^*(R'/K)$ such that
$D^*(d'^*f)(d^*x) = d'^*f(x)$ for $x$ in $R$. It follows that $1 \otimes D^*$ carries $M$
into $D^*(R'/K)$ by $(1 \otimes D^*)(x \otimes y) = xD^*(y)$ with $(1 \otimes D^*)(dz) = 0$
for $z$ in $A$. There is thus a map $E^*$ from $M$ to $D^*(R'/K)$ and one need
only check that $H^*E^*$ and $E^*H^*$ are the identity.

**Lemma 5.** Let $R = K[[X_1, \ldots, X_n]]$ with $[K; K^p] < \infty$ where $p$
is the characteristic exponent of $K$, suppose $A$ is an ideal of $R$ and assume
that $D(A) < A$ for every $K^p$ derivation of $R$ into $R$. If $A \neq 0$ then (i) there
is an element $x$ of $R$ such that $x$ is not in $A$ but $x^p$ is in $A$, or $A = R$
and (ii) there is an element $x$ of the form ring of $R/A$ with $x \neq 0$ and $x^p = 0$
or $A = R$.

**Proof.** Choose a $p$-basis for $K$ over $K^p$ say $y_1, \ldots, y_r$. If $Q$ is a
power series in $R$ we define the total degree of $Q$ to be the pair $(u, v)$
where $u$ is the subdegree of $Q$ and $v$ is the degree of the leading form
of $Q$ considered as a polynomial in the $y_r$. Order the total degrees
lexicographically and choose an element $P$ of $A$ of least total degree
$(a, b)$ and assume $b \neq 0$. Since $b$ is nonzero the partial derivative of $P$
with respect to one of the $y_j$ occurring in the leading form $L(P)$ of $P$ lies in $A$ and reduces the total degree, thus no $y_j$ may occur in $L(P)$ and $L(P)$ is in $K^p[X_1, \cdots, X_n]$. Now consider any one of the indeterminates $X_i$ and note that the subdegree of $P$ will be reduced by differentiating with respect to $X_i$ unless the exponent to which $X_i$ occurs in a given monomial of $L(P)$ is of the form $sp$. We therefore have assertion (ii). If $Q$ is in $A$ we may write it in the form $Q = \sum Q_a$ where $a$ ranges over the collection of all the subsets of $T = \{1, \cdots, n\}$ (including the empty set) and $Q_a$ is the sum of those monomials $M$ of $Q$ such that $X_i$ appears in $M$ with exponent of the form $sp$ for those and only those $i$ in $a$. We now denote by $B_i$ the operation $X_i \partial / \partial X_i$ and note that $B_i$ maps $A$ into itself and that $B_i$ is zero on the monomials of $A$ in which $X_i$ occurs with an exponent which is a multiple of $p$. The application of $B_i (p-1)$ times is the identity on any monomial which does not have $X_i$ occurring with exponent a multiple of $p$. If $Q$ is in $A$ then applying $B_i (p-1)$ times and subtracting the result from $Q$ we have that $\sum Q_a$ is in $A$ where the sum runs over those subsets of $T$ which contain $n$ and by induction we have that $Q_T$ is in $A$. Using (i) we have that there are elements of $A$ such that $Q_T \neq 0$. Let $W$ denote the collection of elements of $A$ of least subdegree which lie in $K[[X_1^p, \cdots, X_n^p]]$ and let $t$ be the least degree of the elements of $W$ considered as polynomials in the $y_j$ with coefficients in $K[[X_1^p, \cdots, X_n^p]]$. To prove assertion (i) it suffices to show that $t=0$. Note, however, that the set $W$ remains fixed under the partial derivatives with respect to the $y_j$ from which the assertion is immediate.

**Theorem.** If $R$ is a local ring containing a field $k$ of characteristic exponent $p$ such that $D^*(R/k^p)$ is finitely generated then the following conditions are equivalent: (i) $A$ is regular, (ii) $D^*(R/k^p)$ is free and if $x$ is an element of $Co(R)$ such that $x^p = 0$ then $x = 0$, (iii) $D^*(R/k^p)$ is free and if $x$ is an element of the form ring of $R$ such that $x^p = 0$ then $x = 0$.

**Proof.** We first remark that if $k < K$ then in characteristic $p$ (nonzero) we have that $D^*(R/k^p) = D^*(R/K^p)$ and in any case by Lemma 3 we may suppose that the ring $R$ is complete. We may therefore assume in nonzero characteristic that $k = K$ is a field of coefficients of $R$. Consider the map $1 \otimes d\#$ carrying the module $m/m^2$ into $R/m \otimes D^*(R/K)$. To prove $1 \otimes d\#$ is an injection it suffices to prove the assertion for $R/m^2 = R^* = K + m^*$ where $m^*$ is the maximal ideal of $R^*$. The projection $g$ of $R^*$ onto $m^*$ is a $K$ derivation to a finitely generated $R^*$ module thus $D^*(g)$ maps $D^*(R/K)$ to $m^*$ such that $D^*(g)d\#m = g(m) = m$ for $m$ in $m^*$, thus $1 \otimes d\#$ is an injection. The map
1 \otimes d^* from m/m^2 to R/m \otimes D^*(R/K^p) is such that if x is in m and 1 \otimes d^*(x) = 0 then 1 \otimes D^*(d\#)(1 \otimes d^*)(x) = 0 which implies x is in m^2 by the above. Thus we have an exact sequence 0 \rightarrow m/m^2 \rightarrow R/m \otimes D^*(R/K^p) \rightarrow D^*(K/K^p) \rightarrow 0 in nonzero characteristic. Similarly in zero characteristic we may replace K^p by k in the above sequence. In nonzero characteristic we have that \([K: K^p] < \infty\) and a basis for \(D(K/K^p)\) is given by the \(dY_j\) where the \(Y_j\) are a \(p\)-basis for \(K\) over \(K^p\), and thus if \(m_i, 1 \leq i \leq n\) is a minimal system of generators for the maximal ideal of \(R\) the elements \(d^*m_j\) and \(d^*Y_i\) are a basis for the module \(D^*(R/K^p)\). In characteristic zero the \(d^*m_j\) are a subset of a basis for \(D^*(R/k)\). Let \(f\) be a map from \(K[[X_1, \cdots, X_n]]\) onto \(R\) carrying \(K\) onto \(K\) and \(X_i\) onto \(X_i\) where the \(X_i\) are indeterminates. Set \(N = \text{kernel} (f)\) and assume \(x\) is in \(N\). We have the equation 0 = \(d^*(f(x)) = \sum f(\partial x/\partial X_i)d^*m_i + \sum f(\partial x/\partial Y_j)d^*Y_j\) in characteristic \(p \neq 0\) and since \(D^*(R/K^p)\) is free all the partials of \(x\) must be in \(N\) which implies \(N = 0\) under the assumptions (ii) and (iii) by Lemma 5. In the case of characteristic zero we have that the partials with respect to the \(X_i\) all are in \(N\) since the \(d^*m_i\) can be extended to a basis and we may again apply Lemma 5.

**Bibliography**