THE RADIUS OF UNIVALENCE OF CERTAIN ANALYTIC FUNCTIONS

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1. Introduction. Suppose that \( f(z) = z + a_2 z^2 + \cdots \) is analytic for \( |z| < 1 \). If \( \Re \{ f(z)/z \} > 0 \) for \( |z| < 1 \) then \( f(z) \) is univalent in \( |z| < \sqrt{2} - 1 \) \cite[Theorem 3; 7]{5}. The function \( f(z) = (z + z^2)/(1 - z) \) satisfies the hypotheses but is univalent in no circle \( |z| < r \) for \( r > \sqrt{2} - 1 \) since its derivative vanishes at \( z = \sqrt{2} - 1 \).

In this paper we generalize the above theorem for functions whose power series begins \( f(z) = z + a_n z^n + \cdots \). The estimate used to obtain this result is further used to find the radius of convexity for functions \( f(z) = z + a_n z^{n+1} + \cdots \) which are analytic and satisfy \( \Re f'(z) > 0 \) for \( |z| < 1 \). For \( n = 1 \) this theorem is not new \cite[Theorem 2; 10, p. 284]{5}. The condition \( \Re f'(z) > 0 \) is known to be sufficient for the univalency of \( f(z) \) in \( |z| < 1 \) \cite[p. 18]{1}.

We consider the problem of finding the radius of univalence for functions \( f(z) = z + a_2 z^2 + \cdots \) which are analytic and satisfy \( \Re \{ f(z)/g(z) \} > 0 \) for \( |z| < 1 \), where \( g(z) = z + b_2 z^2 + \cdots \) is analytic and univalent for \( |z| < 1 \). In the case that \( g(z) \) is either starlike or convex this problem is solved. We take particular advantage of the condition \( \Re \{ z f'(z)/f(z) \} > 0 \) for \( |z| < r \), which is necessary and sufficient for \( f(z) \) to be univalent and starlike in \( |z| < r \) \cite[p. 105, problem 109]{8}. For arbitrary univalent functions \( g(z) \) we only obtain an estimate for the radius of univalence for \( f(z) \).

2. Lemma 1. Suppose that \( h(z) = 1 + c_n z^n + \cdots \) is analytic and satisfies \( \Re h(z) > 0 \) for \( |z| < 1 \). Then
\[
\left| \frac{h'(z)}{h(z)} \right| \leq \frac{2n}{1 - |z|^{2n}}.
\]

Proof. Let \( k(z) = (1 - h(z))/(1 + h(z)) = d_n z^n + \cdots \). Then \( k(z) \) is analytic for \( |z| < 1 \) and \( |k(z)| < 1 \). Thus, \( k(z) = z^\phi(z) \) where \( \phi(z) \) is analytic for \( |z| < 1 \) and \( |\phi(z)| \leq 1 \). For such functions we have
\[
| \phi'(z) | \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}
\]
\[\text{[2, p. 18].}\]
Expressing \( h(z) \) and \( h'(z) \) in terms of \( \phi(z) \) gives

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Using (1) we obtain
\[
\left| \frac{h'(z)}{h(z)} \right| \leq 2 \left| z \right|^{n-1} \left(1 - \left| \phi(z) \right|^2\right) + n \left(1 - \left| z \right|^2\right) \left| \phi(z) \right|.
\]

1To prove the lemma it is sufficient to show that for \(|z|=r, 0<r<1,\)
\[
\frac{r(1 - \left| \phi(z) \right|^2) + n(1 - r^2) \left| \phi(z) \right|}{1 - r^{2n}} \leq \frac{n(1 - r^2)}{1 - r^{2n}}.
\]

Letting \(x = |\phi(z)|\) this is equivalent to \((1-x)F_n(x) \geq 0\) for \(0 \leq x \leq 1,\)
where
\[
F_n(x) = a - bx, \quad a = n(1 - r^2) - r(1 - r^{2n}) > (1 - r^2)(n - nr) > 0,
\]
\[
b = r(1 - r^{2n}) - nr^{2n}(1 - r^2) = r(1 - r^2)(1 + r^2 + r^4 + \cdots + r^{2n-2} - nr^{2n-1})
\]
\[
= r(1 - r^2) \left\{ (1 - r^{2n-1}) + (r^2 - r^{2n-1}) + \cdots + (r^{2n-2} - r^{2n-1}) \right\} > 0.
\]

Since \(F_n(x) \geq F_n(1)\) we can prove \(F_n(x) \geq 0\) by showing that
\[
F_{n+1}(1) \geq F_n(1) = F_1(1) \geq 0.
\]

\[
F_{n+1}(1) - F_n(1) = (1 - r^2)(1 - r^{2n+1} - r^{2n+2} - nr^{2n+2})
\]
\[
= (1 - r^2)(1 + r + r^2 + \cdots + r^{2n} - r^{2n+1} - nr^{2n} - nr^{2n+1})
\]
\[
> 0.
\]

This inequality follows since the negative terms in the brackets can be expressed as \(2n+1\) terms each of which is numerically less than a corresponding positive term.

Finally, \(F_1(1) = (1 + r)(1 - r)^2 > 0.\)

One can show that the equality holds in the lemma only for the functions \(h(z) = \frac{1 - ez^n}{1 + ez^n}\) where \(|e| = 1\) and for appropriate values of \(z.\)

**Theorem 1.** Suppose that \(f(z) = z + a_{n+1}z^{n+1} + \cdots\) is analytic and

\[1\] I would like to thank the referee of this paper for simplifying my argument for the remaining part of the proof.
satisfies $\Re \left\{ \frac{f(z)}{z} \right\} > 0$ for $|z| < 1$. Then $f(z)$ is univalent and starlike in $|z| < ((n^2 + 1)^{1/2} - n)^{1/n}$.

**Proof.** Since $\Re \left\{ \frac{f(z)}{z} \right\} > 0$ we can infer that $f(z)$ cannot vanish in $|z| < 1$ except for a simple zero at $z = 0$. Let

$$h(z) = \frac{f(z)}{z} = 1 + a_{n+1}z^n + \cdots, \quad \Re h(z) > 0 \quad \text{for} \quad |z| < 1.$$  

From Lemma 1 we have

$$\left| \frac{zh'(z)}{h(z)} \right| \leq \frac{2n|z|^n}{1 - |z|^{2n}}.$$  

Also

$$\frac{zf''(z)}{f'(z)} = 1 + \frac{zh'(z)}{h(z)}.$$  

Therefore, $f(z)$ will be univalent and starlike if $|zh'(z)/h(z)| < 1$. From the above estimate this is satisfied if $(2n|z|^n)/(1 - |z|^{2n}) < 1$, i.e., for $|z| < ((n^2 + 1)^{1/2} - n)^{1/n}$.

The function $f(z) = (z + a_{n+1}z^n + \cdots)$ satisfies

$$f(z) = \frac{f(z)}{z} > 0 \quad \text{for} \quad |z| < 1$$

but is not univalent in $|z| < r$ for $r > r_n = ((n^2 + 1)^{1/2} - n)^{1/n}$ since $f'(r_n e^{i(\pi/n)}) = 0$.

**Theorem 2.** Suppose that $f(z) = z + a_{n+1}z^n + \cdots$ is analytic and satisfies $\Re f'(z) > 0$ for $|z| < 1$. Then $f(z)$ is convex in $|z| < ((n^2 + 1)^{1/2} - n)^{1/n}$.

**Proof.** We can apply Lemma 1 to $f'(z) = 1 + (n+1)a_{n+1}z^n + \cdots$ since $\Re f'(z) > 0$. This gives

$$\left| \frac{f''(z)}{f'(z)} \right| \leq 2n|z|^{n-1}/1 - |z|^{2n}.$$  

The condition $\Re \left\{ (zf''(z)/f'(z)) + 1 \right\} > 0$ for $|z| < r$ is necessary and sufficient for $f(z)$ to map $|z| < r$ onto a convex domain [8, problem 108, p. 105]. This condition is satisfied if $|zf''(z)/f'(z)| < 1$. From the above estimate we can deduce that $f(z)$ is convex if $(2n|z|^n)/(1 - |z|^{2n}) < 1$. This inequality is equivalent to $|z| < ((n^2 + 1)^{1/2} - n)^{1/n}$.

The function

$$f(z) = \int_0^z \frac{1 + \sigma^n}{1 - \sigma^n} d\sigma = z + \frac{2}{n+1} z^{n+1} + \cdots$$

is an extremal function for Theorem 2.
3. **Theorem 3.** Suppose that $f(z) = z + a_2z^2 + \cdots$ and $g(z) = z + b_2z^2 + \cdots$ are analytic for $|z| < 1$ and $g(z)$ is univalent and starlike for $|z| < 1$. If $\text{Re} \left\{ f(z)/g(z) \right\} > 0$ for $|z| < 1$ then $f(z)$ is univalent and starlike in $|z| < 2 - \sqrt{3}$.

**Proof.** The hypotheses imply that $f(z)$ and $g(z)$ do not vanish in $|z| < 1$ except for the simple zero at $z = 0$. Let

$$h(z) = \frac{f(z)}{g(z)} = 1 + c_1z + \cdots, \quad \text{Re} \ h(z) > 0 \quad \text{for} \quad |z| < 1.$$ 

Applying Lemma 1 to $h(z)$ for $n = 1$ gives $|zh'(z)/h(z)| \leq 2|z|/(1 - |z|^2)$. Since $g(z)$ is starlike $\text{Re} \{ zg'(z)/g(z) \} > 0$ for $|z| < 1$. Thus $\text{Re} \{ zg'(z)/g(z) \} \geq (1 - |z|)/(1 + |z|)$ [8, problem 287, p. 140].

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} = \text{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} - \frac{|zh'(z)|}{h(z)} \geq \frac{1 - |z|}{1 + |z|} - \frac{2|z|}{1 - |z|^2} = \frac{1 - 4|z| + |z|^2}{1 - |z|^2}.$$ 

Thus, $\text{Re} \left\{ zf'(z)/f(z) \right\} > 0$ if $1 - 4|z| + |z|^2 > 0$. The last inequality is satisfied for $|z| < 2 - \sqrt{3}$. Therefore $f(z)$ is univalent in $|z| < 2 - \sqrt{3}$ and maps that circle onto a starlike domain.

The function $f(z) = (z + z^2)/(1 - z)^3$ satisfies the hypotheses of Theorem 3 where $g(z) = z/(1 - z)^2$ and $h(z) = (1 + z)/(1 - z)$. The derivative of this function vanishes at $z = -2/\sqrt{3}$. Thus, it is univalent in no circle $|z| < r$ with $r > 2 - \sqrt{3}$.

For a part of the next theorem we need a sharpening of Lemma 1 for $n = 1$. This result is known but we give a short proof of it here.

**Lemma 2.** Suppose that $h(z) = 1 + c_1z + \cdots$ is analytic and satisfies $\text{Re} \ h(z) > 0$ for $|z| < 1$. Then $|h'(z)| \leq 2 \text{ Re} \ h(z)/(1 - |z|^2)$.

**Proof.** Let $\phi(z) = (1 - h(z))/(1 + h(z))$, $|\phi(z)| < 1$ for $|z| < 1$. Using

$$|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2},$$

(1)

The author thanks the referee for indicating that the proof of this lemma can be obtained so readily from the estimate (1).
The lemma follows by noting that $\left| 1 + h(z) \right|^2 - \left| 1 - h(z) \right|^2 = 4 \text{Re} h(z)$.

**Theorem 4.** Suppose that $f(z) = z + a_2 z^2 + \cdots$ and $g(z) = z + b_2 z^2 + \cdots$ are analytic for $|z| < 1$ and $g(z)$ is univalent and convex for $|z| < 1$. If $\text{Re} \left\{ f(z)/g(z) \right\} > 0$ for $|z| < 1$ then $\text{Re} \left\{ f'(z)/g'(z) \right\} > 0$ for $|z| < \frac{1}{2}$. Also, $f(z)$ is univalent and starlike for $|z| < \frac{1}{3}$.

**Proof.** The hypotheses imply that $f(z)$ and $g(z)$ do not vanish in $|z| < 1$ except for the simple zeros of $f(z)$ and $g(z)$ at $z = 0$. Let $h(z) = f(z)/g(z) = 1 - \frac{a_2}{b_2} z + \cdots$, $\text{Re} h(z) > 0$ for $|z| < 1$.

Applying Lemma 2 to $h(z)$ gives $|h'(z)| \leq 2 \text{Re} h(z)/(1 - |z|^2)$. Since $g(z)$ is univalent and convex for $|z| < 1$ we have $\text{Re} \left\{ z g'(z)/g(z) \right\} > 0$ for $|z| < 1$ and consequently $\text{Re} \left\{ z g'(z)/g(z) \right\} \geq (1 + |z|)^{-1} \left[ 6; 9 \right]$. This implies $|g(z)/g'(z)| \geq |z| (1 + |z|)$.

Thus, for $|z| < \frac{1}{3}$ $\text{Re} \left\{ f'(z)/g'(z) \right\} > 0$. This shows that $f(z)$ is univalent and close-to-convex for $|z| < \frac{1}{3}$ [4].

Let us show that $f(z)$ maps $|z| < \frac{1}{3}$ onto a starlike domain.

$$\frac{z f'(z)}{f(z)} = \frac{z g'(z)}{g(z)} + \frac{z h'(z)}{h(z)}$$

$$\text{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} \geq \text{Re} \left\{ \frac{z g'(z)}{g(z)} \right\} - \frac{z h'(z)}{h(z)}$$

$$\geq \frac{1}{1 + |z|} - 2 \frac{|z|}{1 - |z|^2} = \frac{1 - 3 |z|}{1 - |z|^2}.$$

For $|z| < \frac{1}{3}$ $\text{Re} \left\{ z f'(z)/f(z) \right\} > 0$. Thus, $f(z)$ is starlike in $|z| < \frac{1}{3}$.

Theorem 4 gives the radius of univalence for the class of functions considered. In order to show this let $f(z) = (z + z^2)/(1 - z)^2$, $g(z)$
= z/(1 − z). Then, g(z) is univalent and convex for |z| < 1. Here, h(z) = (1 + z)/(1 − z) and therefore Re h(z) > 0. This function f(z) is univalent in no circle |z| < r with r > \frac{1}{2} since f'(−\frac{1}{2}) = 0.

**Theorem 5.** Suppose that f(z) = z + a_2z^2 + \cdots and g(z) = z + b_2z^2 + \cdots are analytic for |z| < 1 and g(z) is univalent in |z| < 1. If Re {f(z)/g(z)} > 0 for |z| < 1 then f(z) is univalent in |z| < 1/5.

**Proof.** Let h(z) = f(z)/g(z) = 1 + cz + \cdots, Re h(z) > 0 for |z| < 1.

To show that f(z) is univalent in |z| ≤ r it suffices to show that f(z) is univalent on |z| = r. Let z_1 \neq z_2, |z_1| = |z_2| = r. Then, f(z_1) = f(z_2) can be written

\[
\frac{1}{g(z_1)} \frac{g(z_2) - g(z_1)}{z_2 - z_1} = - \frac{1}{h(z_2)} \frac{h(z_2) - h(z_1)}{z_2 - z_1}.
\]

Thus, if

\[
\left| \frac{g(z_2) - g(z_1)}{g(z_1)(z_2 - z_1)} \right| > \left| \frac{h(z_2) - h(z_1)}{h(z_2)(z_2 - z_1)} \right|
\]

then f(z) is univalent in |z| ≤ r.

Let k(z) = (1 − h(z))/(1 + h(z)), |k(z)| < 1 for |z| < 1 and k(0) = 0. Therefore |k'(z)| ≤ 1 for |z| ≤ \sqrt{2} − 1 [2, p. 19]. From the representation k(z_2) − k(z_1) = \int_{z_1}^{z_2} k'(z)dz where the path of integration is the line segment from z_1 to z_2 the estimate on k'(z) gives \left| \frac{(k(z_2) − k(z_1))/(z_2 − z_1)} \right| ≤ 1 for r ≤ \sqrt{2} − 1. Expressing h(z) in terms of k(z) yields

\[
\frac{k(z_2) - k(z_1)}{h(z_2)(z_2 - z_1)} = - \frac{2k(z_2) - k(z_1)}{z_2 - z_1} \frac{1}{(1 + k(z_1))(1 - k(z_2))}
\]

\[
\left| \frac{k(z_2) - k(z_1)}{h(z_2)(z_2 - z_1)} \right| \leq \frac{2}{(1 - |z_1|)(1 - |z_2|)} = \frac{1 - r^2}{(1 - r)^2}.
\]

Here we have used Schwarz’s lemma |k(z)| ≤ |z|.

Since g(z) = z + b_2z^2 + \cdots is analytic and univalent for |z| < 1

\[
\left| \frac{g(z_2) - g(z_1)}{z_2 - z_1} \right| \geq \left| g(z_1)g(z_2) \right| \frac{1 - r^2}{r^2}
\]

[3]. Using this estimate and the distortion theorem \left| g(z) \right| \geq |z|/(1 + |z|)^2 we obtain
Therefore, \( f(z) \) is univalent in \( |z| \leq r \) if \( r \leq \sqrt{2} - 1 \) and \((1-r)/(r(1+r)) > 2/(1-r)^2\). The last inequality is equivalent to \( 1 - 5r + r^2 - r^3 > 0 \). Since the equation \( 1 - 5r + r^2 - r^3 = 0 \) has one positive root \( r_0 \), where \( 0.20 < r_0 < 0.21 \), we can infer that \( f(z) \) is univalent in \( |z| < r_0 \). In particular, \( f(z) \) is univalent in \( |z| < 1/5 \).

The circle \( |z| < r_0 \) is not the circle of univalence for the functions \( f(z) \) which satisfy the hypotheses of Theorem 5. If it were then we must have \( |g(z)| = |z|/(1 + |z|)^2 \) for some \( z \). This estimate holds only for the functions \( g(z) = z/(1 + \varepsilon z)^2 \) where \( |\varepsilon| = 1 \). Since these functions are starlike for \( |z| < 1 \) Theorem 3 implies that \( f(z) \) would be univalent \( |z| < 2 - \sqrt{3} \). However, \( 2 - \sqrt{3} = 0.267 \cdots \) > \( r_0 \).

**References**


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