ON THE QUOTIENT OF ENTIRE FUNCTIONS OF LOWER ORDER LESS THAN ONE

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The origin of this note is a question in the theory of functions: “If \( f_1(z) \) and \( f_2(z) \) are two entire functions of lower order less than one and if \( f_1(z) \) and \( f_2(z) \) have the same zeros, is \( f_1(z)/f_2(z) \) a constant?” This is one of 25 problems published in Bulletin of the American Mathematical Society, January, 1962, pp. 21–24.

The solution of this problem is that the quotient \( f_1(z)/f_2(z) \) is not necessarily a constant. It is even possible to find such entire functions of lower order zero. To do this we introduce some definitions.

\[
\begin{align*}
  a_n &= 2^{4n+1}, \quad b_n = 2^{4n+2}, \\
  P_n(z) &= \left( 1 - \frac{z}{a_n} \right) \left( 1 + \frac{z}{b_n} \right), \\
  f_1(z) &= \prod_{n=1}^{\infty} P_n(z), \quad f_2(z) = e^{-f_1(z)}. 
\end{align*}
\]

Now \( f_1(z) \) and \( f_2(z) \) are different entire functions with the same zeros. We denote

\[
M_\nu(r) = \max_{|z|=r} |f_\nu(z)|, \quad \nu = 1, 2.
\]

We shall prove that the lower order of each of these functions is zero i.e.

\[
\liminf_{r \to \infty} \frac{\log \log M_\nu(r)}{\log r} = 0, \quad \nu = 1, 2.
\]

We first estimate \( \log \log M_1(r) \) for \( r = 2^{4m+3} \). Obviously, for \( |z| = 2^{4m+3} \) we have

\[
|P_n(z)| = \left| 1 - \frac{z}{a_n} \right| \left| 1 + \frac{z}{b_n} \right| < |z|^{a_n+b_n}
\]

i.e.,

\[
\log |P_n(z)| < \log 2 \cdot (4m + 3)! (a_n + b_n)
\]

which implies

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\[
\log | P_1(z) \cdot \cdots \cdot P_m(z) | < \log 2 \cdot (4m + 3)! (a_1 + b_1 + a_2 + b_2 + \cdots + a_m + b_m).
\]

Roughly estimated
\[
\log | P_1(z) \cdot \cdots \cdot P_m(z) | < (4m + 3)! b_m = (4m + 3)! 2^{(4m+3)!}.
\]

For \( P_{m+n}(z) \), \( n \geq 1 \), we use another estimate. The following simple inequalities are well known
\[
| \log \left( 1 - \frac{z}{a} \right)^a + z | \leq \frac{| z |^2}{a} \quad \text{for} \quad | z | < \frac{a}{2},
\]
\[
| \log \left( 1 + \frac{z}{b} \right)^b - z | \leq \frac{| z |^2}{b} \quad \text{for} \quad | z | < \frac{b}{2}.
\]

With \( a = a_{m+n} \) and \( b = b_{m+n} \) the estimate becomes
\[
\log | P_{m+n}(z) | \leq \log | P_{m+n}(z) | = \left| \log \left( 1 - \frac{z}{a_{m+n}} \right)^{a_{m+n}} + \log \left( 1 - \frac{z}{b_{m+n}} \right)^{b_{m+n}} \right|
\]
\[
\leq \left| \log \left( 1 - \frac{z}{a_{m+n}} \right)^{a_{m+n}} + \log \left( 1 + \frac{z}{b_{m+n}} \right)^{b_{m+n}} - z \right|
\]
\[
\leq \frac{| z |^2}{a_{m+n}} + \frac{| z |^2}{b_{m+n}} < 2 \left( \frac{| z |^2}{a_{m+n}} \right) = 2^{1 + 2(4m+3)! - (4m+4)!} < 2^{-n}.
\]

Thus the infinite product
\[ P_{m+1}(z) \cdot P_{m+2}(z) \cdot \cdots \cdot P_{m+n}(z) \cdot \cdots \]

is estimated by
\[
\log | P_{m+1}(z) \cdot P_{m+2}(z) \cdot \cdots \cdot P_{m+n}(z) | < \sum_{n=1}^{\infty} 2^{-n} = 1.
\]

For
\[
M_1(r) = \max_{|z|=r} \left| \prod_{n=1}^{\infty} P_n(z) \right|
\]
we then get
\[
\log M_1(r) < (4m + 3)! 2^{(4m+3)!} + 1 < 2^{2 \cdot (4m+3)!}
\]
\[
\log \log M_1(r) < 2 \log 2 \cdot (4m + 2)!
\]

where \( \log r = \log 2 \cdot (4m+3)! \). Thus
\[
\frac{\log \log M_1(r)}{\log r} < \frac{2}{4m + 3} < \frac{1}{m}.
\]

This estimate implies that the lower order of \(f_1(z)\) is zero. We now consider \(\log M_2(r)\) for \(r = 2^{(4m+1)!}\) and in the same way as before we obtain

\[
\log \left| P_1(z) \cdot P_2(z) \cdot \cdots \cdot P_{m-1}(z) \right| < \log 2 \cdot (4m + 1)! (a_1 + b_1 + \cdots + a_{m-1} + b_{m-1})
\]

\[
< (4m + 1)! b_{m-1} = (4m + 1)! 2^{(4m - 2)!}.
\]

Then we consider \(P_m(z) \cdot e^{-z}\). We have

\[
\log \left| P_m(z) e^{-z} \right| \leq \log \left| 1 - \frac{z}{a_m} \right|^{a_m} + \log \left| 1 + \frac{z}{b_m} \right|^{b_m} - z \leq \log \left| z \right| a_m + \frac{|z|^2}{b_m} = \log 2(4m + 1)! 2^{(4m)!} + 2^2 \cdot (4m + 1)! 2^{(4m - 2)!} \\
< (4m + 1)! 2^{(4m)!} + 1.
\]

For \(P_{m+n}(z)\) we obtain as before

\[
\log \left| P_{m+n}(z) \right| < 2^{-n}, \quad n \geq 1.
\]

Hence

\[
\log \left| P_{m+1}(z) \cdot P_{m+2}(z) \cdots \right| < 1.
\]

For

\[
M_2(r) = \max_{|z| = r} \left| P_1(z) \cdot \cdots \cdot P_{m-1}(z) \cdot P_m(z) \cdot e^{-z} \cdot P_{m+1}(z) \cdots \right|
\]

we obtain

\[
\log M_2(r) < (4m + 1)! 2^{(4m - 2)!} + (4m + 1)! 2^{(4m)!} + 1 + 1 < 2^{2 \cdot (4m)!}.
\]

Now \(\log \log M_2(r) < 2 \log 2 \cdot (4m)!\) and \(\log r = \log 2 \cdot (4m + 1)!\). Thus

\[
\frac{\log \log M_2(r)}{\log r} < \frac{2}{4m + 1} < \frac{1}{m}
\]

which implies that the lower order of \(f_2(z)\) is zero. The proof is now complete.

Kungl. Tekniska Högskolan, Stockholm