ON THE QUOTIENT OF ENTIRE FUNCTIONS OF
LOWER ORDER LESS THAN ONE

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The origin of this note is a question in the theory of functions: "If
f_1(z) and f_2(z) are two entire functions of lower order less than one and
if f_1(z) and f_2(z) have the same zeros, is f_1(z)/f_2(z) a constant?" This
is one of 25 problems published in Bulletin of the American Mathem-

The solution of this problem is that the quotient f_1(z)/f_2(z) is not
necessarily a constant. It is even possible to find such entire functions
of lower order zero. To do this we introduce some definitions.

\[ a_n = 2^{(4n)!}, \quad b_n = 2^{(4n+2)!}, \]

\[ P_n(z) = \left(1 - \frac{z}{a_n}\right)^{a_n} \left(1 + \frac{z}{b_n}\right)^{b_n}, \]

\[ f_1(z) = \prod_{n=1}^{\infty} P_n(z), \quad f_2(z) = e^{-f_1(z)}. \]

Now f_1(z) and f_2(z) are different entire functions with the same zeros.
We denote

\[ M_v(r) = \max_{|z| = r} |f_v(z)|, \quad v = 1, 2. \]

We shall prove that the lower order of each of these functions is zero
i.e.

\[ \liminf_{x \to \infty} \frac{\log \log M_v(r)}{\log r} = 0, \quad v = 1, 2. \]

We first estimate \log \log M_1(r) for \( r = 2^{(4m+3)!} \). Obviously, for \( |z| = 2^{(4m+3)!} \) we have

\[ |P_n(z)| = \left|1 - \frac{z}{a_n}\right|^{a_n} \left|1 + \frac{z}{b_n}\right|^{b_n} < |z|^{a_n+b_n} \]

i.e.,

\[ \log |P_n(z)| < \log 2 \cdot (4m + 3)!(a_n + b_n) \]

which implies

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\[
\log | P_1(z) \cdot \ldots \cdot P_m(z) | \\
< \log 2 \cdot (4m + 3)! \left( a_1 + b_1 + a_2 + b_2 + \ldots + a_m + b_m \right).
\]

Roughly estimated
\[
\log | P_1(z) \cdot \ldots \cdot P_m(z) | < (4m + 3)! b_m = (4m + 3)!2^{(4m+3)}!.
\]

For \( P_{m+n}(z) \), \( n \geq 1 \), we use another estimate. The following simple inequalities are well known
\[
\log (1 - \frac{z}{a})^a + z \leq \frac{|z|^2}{a} \quad \text{for} \quad |z| < \frac{a}{2},
\]
\[
\log (1 + \frac{z}{b})^b - z \leq \frac{|z|^2}{b} \quad \text{for} \quad |z| < \frac{b}{2}.
\]

With \( a = a_{m+n} \) and \( b = b_{m+n} \) the estimate becomes
\[
\log | P_{m+n}(z) | \leq \log P_{m+n}(z)
\]
\[
= \left| \log \left( 1 - \frac{z}{a_{m+n}} \right)^{a_{m+n}} + \log \left( 1 - \frac{z}{b_{m+n}} \right)^{b_{m+n}} - z \right|
\]
\[
\leq \left| \log \left( 1 - \frac{z}{a_{m+n}} \right)^{a_{m+n}} + \log \left( 1 + \frac{z}{b_{m+n}} \right)^{b_{m+n}} - z \right|
\]
\[
\leq \frac{|z|^2}{a_{m+n}} + \frac{|z|^2}{b_{m+n}} < \frac{2 |z|^2}{a_{m+n}} = 2^{1+2(4m+3)!-(4m+4n)!} < 2^{-n}.
\]

Thus the infinite product
\[
P_{m+1}(z) \cdot P_{m+2}(z) \cdot \ldots \cdot P_{m+n}(z) \cdot \ldots
\]

is estimated by
\[
\log | P_{m+1}(z) \cdot P_{m+2}(z) \cdot \ldots \cdot P_{m+n}(z) \cdot \ldots | < \sum_{n=1}^{\infty} 2^{-n} = 1.
\]

For
\[
M_1(r) = \max_{|z|=r} \left| \prod_{n=1}^{\infty} P_n(z) \right|
\]

we then get
\[
\log M_1(r) < (4m + 3)!2^{(4m+3)}! + 1 < 2^{4m+3}!
\]
\[
\log \log M_1(r) < 2 \log 2 \cdot (4m + 2)!
\]

where \( \log r = \log 2 \cdot (4m + 3)! \). Thus
\[
\frac{\log \log M_1(r)}{\log r} < \frac{2}{4m + 3} < \frac{1}{m}.
\]

This estimate implies that the lower order of \(f_1(z)\) is zero. We now consider \(\log \log M_2(r)\) for \(r = 2^{(4m+1)!}\) and in the same way as before we obtain

\[
\log \left| P_1(z) \cdot P_2(z) \cdot \cdots \cdot P_{m-1}(z) \right| < \log 2 \cdot (4m + 1) ! (a_1 + b_1 + \cdots + a_{m-1} + b_{m-1}) < (4m + 1) ! b_{m-1} = (4m + 1) ! 2^{(4m-2)!}.
\]

Then we consider \(P_m(z) \cdot e^{-z}\). We have

\[
\log \left| P_m(z) e^{-z} \right| \leq \log \left| 1 - \frac{z}{a_m} \right|^{a_m} + \log \left( 1 + \frac{z}{b_m} \right)^{b_m} - z
\]

\[
< \log \left| z \right| a_m + \frac{z^2}{b_m} = \log 2(4m + 1) ! 2^{(4m)!} + 2^2 \cdot 2^{(4m+1)!} - (4m+2)! < (4m + 1) ! 2^{(4m)!} + 1.
\]

For \(P_{m+n}(z)\) we obtain as before

\[
\log \left| P_{m+n}(z) \right| < 2^{-n}, \quad n \geq 1.
\]

Hence

\[
\log \left| P_{m+1}(z) \cdot P_{m+2}(z) \cdot \cdots \right| < 1.
\]

For

\[
M_2(r) = \max_{|z| = r} \left| P_1(z) \cdot \cdots \cdot P_{m-1}(z) \cdot P_m(z) \cdot e^{-z} \cdot P_{m+1}(z) \cdot \cdots \right|
\]

we obtain

\[
\log M_2(r) < (4m + 1) ! 2^{(4m-2)!} + (4m + 1) ! 2^{(4m)!} + 1 + 1 < 2^2 \cdot (4m)!.
\]

Now \(\log M_2(r) < 2 \log 2 \cdot (4m)!\) and \(\log r = \log 2 \cdot (4m+1)!\). Thus

\[
\frac{\log \log M_2(r)}{\log r} < \frac{2}{4m + 1} < \frac{1}{m}
\]

which implies that the lower order of \(f_2(z)\) is zero. The proof is now complete.

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