ON RELATIONS AMONG SOME CONSTANTS OF AN ENTIRE FUNCTION

J. GOPALA KRISHNA

Throughout this paper $f(x)$ stands for a function, positive and increasing for real $x \geqq 1$, ρ for a fixed finite positive real, and

$$\begin{aligned} L &= L(\rho, f) \\ l &= l(\rho, f) \end{aligned} \quad \text{for} \lim_{x \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \; x^{-\rho} f(x)$$

$$\begin{aligned} M &= M(\rho, f) \\ m &= m(\rho, f) \end{aligned} \quad \text{for} \lim_{x \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \; x^{-\rho} \int_1^x f(t)/t dt.$$

If $P(z)$ is an entire function of order ρ, it is known that (1) M, m are its type and co-type, in case f happens to be the rank (or position) function of the maximum term of the Maclaurin series of P (see Chapter II of [1] and Chapter II of [2]) and that (2) M and m are the logarithmic type and co-type of P, if $f(x)$ is the number of zeroes z of $P(z)$ such that $|z| \leqq x$ (see Jensen's Theorem, Chapter III of [2]).

Also let $\phi(t)$ and $\psi(t)$ stand for the extended real valued strictly decreasing and continuous functions over the closed (0, 1), defined by the equations (see Lemmas I and II of [3])

$$(1 - \phi(t)) \exp \phi(t) = t,$$

$$(1 + \psi(t)) \exp(-\psi(t)) = t.$$

C. R. Rao, improving upon a number of results known earlier, has shown (see [3] and its corrigendum),

THEOREM I′. *For an f*

(1) $l \leqq \rho m \leqq \rho M \leqq L$,

(2) $L \leqq \operatorname{Min}[\rho M \exp (1 - m\rho/L, \rho M \exp \phi(m/M)]$,

(3) $m\rho/[1+\psi(m/M)] \leqq m\rho/[1+\log(M\rho/l)] \leqq l$,

with certain conventions such as $m/M = 0$ if either of m, M is 0 or $+\infty$ (explicitly stated in [4]), so arranged as to make the group (1), (2), (3) *include the statements*

(4) '$L = +\infty \leftrightarrow M = +\infty$',

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(5) '$L=0 \leftrightarrow M=0$', and

(6) '*In the presence of* $M<+\infty$, $l=0 \rightarrow m=0$'.

The genesis of this paper lies in an attempt to obtain a complete class of independent relations between the four constants L, M, m, l of an f. The lemma proved in this paper (useful in a number of other contexts not considered here) enables one to find all classes of independent relations in the case of several relations between nonnegative reals, expressed by (1), (2) and (3), brings out the relation between ϕ and ψ, and leads to Theorems I and II. I prove a more precise form of (2) and (3):

THEOREM I. *For an f, with the conventions of Theorem* I′,

(i) $L \leqq \rho M \exp(1-m\rho/L) \leqq \rho M \exp \phi(m/M)$,

and further this holds with only one of $<$ *or* $=$ *throughout, except when* $m=0$ *and* $L<e\rho M$.

(ii) $m\rho/[1+\psi(m/M)] \leqq m\rho/[1+\log(M\rho/l) \leqq l$,

holds with only one of $<$ *or* $=$ *throughout, except when* $l>0$ *and* $M=+\infty$.

C. R. Rao deduces from (2) of Theorem I′,

THEOREM II′. *For an entire function of order ρ and of positive finite type τ, for which f is the counting function of zeros* [1]

$$L \leqq \rho\tau \operatorname{Min}[\exp(1-l/L), \exp \phi(l/\rho\tau)],$$

while in an earlier paper [5], *S. M. Shah has shown that*

$$L \leqq \rho\tau \exp(1-l/L).$$

Here I prove

THEOREM II. *For an entire function of order ρ and type τ for which f is the counting function of zeroes,*

$$L \leqq \rho\tau \exp(1-l/L) \leqq \rho\tau \exp \phi(l/\rho\tau)$$

(with the additional convention that $l/\rho\tau=0$, if either of l or τ is 0 or $+\infty$) and further this holds with only one of $<$ *or* $=$ *throughout, except in the case $l=0$ and $L<e\rho\tau$, and in case $L<+\infty$ and $\tau=+\infty$.*

The discussion of a complete class of relations between the four constants of an f for which $\limsup_{x\to+\infty} \log f(x)/\log x=\rho$ is of interest in connection with the study of an entire function of order ρ. This is lengthy and I postpone it.

LEMMA. *Let R stand throughout for any chosen one of the relations $<$, $=$ or $>$. Let $0<a\leqq b<+\infty$, and $0<c<+\infty$. Let, for brevity, $\alpha=a/[1+\psi(a/b)]$ and $\beta=b\exp\phi(a/b)$. Then*

(A) (i) *P_1: '$cR\beta$', P_2: '$b\exp(1-a/c)R\beta$', and P_3: '$cRb\exp(1-a/c)$', are equivalent, in case $c\geqq a$;*

(ii) *in case $c<a$, P_1 and P_2 hold with $<$ for R and P_3 also so holds, if and only if further $c>\alpha$; and*

(B) (i) *P_4: 'αRc', P_5: '$\alpha[1+\log(b/c)]Ra$', and P_6: '$aRc[1+\log(b/c)]$', are equivalent, in case $c\leqq b$;*

(ii) *in case $c>b$, P_4 and P_5 hold with $<$ for R and P_6 also so holds, if and only if further $c<\beta$.*

PROOF OF (A)(i). Let ϕ^{-1} stand for the inverse function of ϕ. Since, by the definition of ϕ, $a/b=[1-\phi(a/b)]\exp\phi(a/b)$, we have,

$$P_1\leftrightarrow \text{'}a/bR(a/c)\exp\phi(a/b)\text{'}\leftrightarrow \text{'}1-\phi(a/b)Ra/c\text{'}\leftrightarrow P_2;$$

and on account of the strictly decreasing monotonicity of ϕ, and the fact '$a\leqq c$,' and the definition of ϕ, by which $\phi^{-1}(1-a/c)=[1-(1-a/c)]\exp(1-a/c)$ we have,

$$P_2\leftrightarrow \text{'}1-a/cR\phi(a/b)\text{'}\leftrightarrow \text{'}(a/b)R\phi^{-1}(1-a/c)\text{'}$$

which $\leftrightarrow$'$a/bR(a/c)\exp(1-a/c)$'$\leftrightarrow P_3$.

PROOF OF (A)(ii). Let ψ^{-1} stand for the inverse function of ψ.

The first part of (A) (ii) is obvious.

On account of the strictly decreasing monotonicity of ψ and the fact '$c<a$,' and the definition of ψ, by which $\psi^{-1}(a/c-1)=[1+(a/c-1)]\exp(1-a/c)$, we have,

$$P_3\leftrightarrow \text{'}a/bR(a/c)\exp(1-a/c)\text{'}\leftrightarrow \text{'}a/bR\psi^{-1}(a/c-1)\text{'}$$

which $\leftrightarrow$'$a/c-1R\psi(a/b)$'$\leftrightarrow$'αRc.'

This completes the proof of (A).

Proof of (B) is similar.

PROOF OF THEOREM I. In case $0<m$ and $L<+\infty$, by (1), $0<\rho m\leqq\rho M\leqq L<+\infty$, and hence by (2) and the lemma (A) (taking $a=\rho m$, $b=\rho M$ and $c=L$), follows (i).

In case $L=+\infty$ so is M by (2), and hence (i).

In case $m=0$ by (2), (i) is trivial.

(ii) is similarly proved, by distinguishing the cases $0<l$, $M<+\infty$; $0=l$, $M<+\infty$ (when $m=0$ by (3)); $0=l$, $M=+\infty$; $0<l$, $M=\infty$.

NOTE. On essentially similar lines, we may improve the more general Theorem 3 of [4].

PROOF OF THEOREM II. In virtue of (2.5.17) of [1] viz., $M \leqq \tau$ and (2), according to which L and M are either both 0 or both $+\infty$, we have,

$$\text{`}\tau < +\infty\text{'} \rightarrow \text{`}L < +\infty\text{'}$$
$$\text{`}\tau = 0\text{'} \rightarrow \text{`}L = 0\text{'} \text{ and}$$
$$\text{`}L = +\infty\text{'} \rightarrow \text{`}\tau = +\infty\text{'}.$$

We distinguish the cases $0 < l$, $\tau < +\infty$; $\tau = +\infty$, $L = +\infty$; and $\tau = +\infty$, $L < +\infty$; $l = 0$.

In the first case by the above results and (1), $0 < l \leqq \rho\tau < +\infty$ and $l \leqq L < +\infty$, and hence by Theorem II′ and (A) of the Lemma follows the result. In the other cases the result easily follows from Theorem II′ and the above mentioned facts.

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ANDHRA UNIVERSITY, INDIA