SHORTER NOTE

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

A NEW PROOF OF A STABILITY ESTIMATE OF LYAPUNOV[1]

HARRY HOCHSTADT

A classical theorem due to Lyapunov [**1**; **2**] is the following:

THEOREM. *Let $p(t)\not\equiv 0$ be a non-negative, piecewise continuous, periodic function of period T. Then all solutions of*

$$y'' + p(t)y = 0$$

are bounded for all t if

$$T\int_0^T p(t)dt \leqq 4.$$

The purpose of this note is to prove the following theorem, which implies Lyapunov's theorem.

THEOREM. *Let $p(t)\not\equiv 0$ be a non-negative, integrable, periodic function of period T, and $\bar{\lambda}$ the smallest eigenvalue of the boundary value problem*

$$\begin{aligned} y'' + \lambda p(t)y &= 0, \\ y(0) + y(T) &= 0, \\ y'(0) + y'(T) &= 0. \end{aligned}$$

If $T\int_0^T p(t)dt \leqq 4$ then $\bar{\lambda} > 1$.

PROOF. From the boundary conditions it is clear that the solution y cannot be of constant sign. If $\bar{\lambda}$ is the smallest eigenvalue y must vanish precisely once in the half-open interval $(0, T]$. Also

$$\begin{aligned} y'' &\leqq 0 \qquad \text{for } y > 0 \\ y'' &\geqq 0 \qquad \text{for } y < 0. \end{aligned}$$

Without loss of generality $y(0)\geqq 0$, and we suppose for the moment that $y'(0)\geqq 0$. Since $y''\geqq 0$ the function has nonpositive curvature

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and if we suppose that $y(c)=0$ then y has precisely one maximum in $[0, c]$. In the interval $[c, T]$ $y'\leqq 0$, since $y'(c)<0$ and $y'(T)\leqq 0$ and y' cannot change sign in the interval. This would violate the condition $y''\geqq 0$. The case where $y'(0)<0$ is handled in a similar manner. It follows that the closed interval $[0, T]$ can be split into two subintervals, in one of which y is nondecreasing, and in the other nonincreasing. In extreme cases one of these subintervals could vanish, in which case $|y|$ max must occur on the boundary. Bearing in mind these considerations and the boundary condition $y(0)+y(T)=0$ it follows that

$$4(y^2)_{\max} = \left(\int_0^T |y'|\, dt\right)^2 \leqq T\int_0^T y'^2 dt.$$

Then

$$\int_0^T y'^2 dt = \bar{\lambda}\int_0^T py^2 dt < \bar{\lambda}(y^2)_{\max}\int_0^T p dt \leqq \bar{\lambda}\int_0^T y'^2 dt$$

so that $\bar{\lambda}>1$.

The proof of Lyapunov's theorem is completed by observing that the first stability zone of the Hill's equation is given by $0<\lambda<\bar{\lambda}$. It follows that $\lambda=1$ is in the stable region.

That this result is the best possible has been shown in [**3**]. For $p(t)=(4/T)\delta(t-T/2)$, an impulse function, a calculation shows that $\bar{\lambda}=1$. It is therefore plausible that for every $\epsilon>0$ a sufficiently close approximation to the impulse function, for which $T\int_0^T p(t)dt=4+\epsilon$, can be found for which $\bar{\lambda}=1$.

References

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3. E. R. Van Kampen, and A. Wintner, *On an absolute constant in the theory of variational stability*, Amer. J. Math. (1937), 270–274.

Polytechnic Institute of Brooklyn