A GENERALIZATION OF ABSOLUTE RIESZIAN SUMMABILITY

B. J. BOYER AND L. I. HOLDER

1. Introduction. Absolute Rieszian summability was defined in 1928 by N. Obreschkoff [4; 5] as follows:

Definition 1. Let \( k > 0 \), and \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n, \lambda_n \to \infty \) as \( n \to \infty \). Let

\[
C^k_\lambda(\omega) = \sum_{\lambda_n < \omega} a_n \left( 1 - \frac{\lambda_n}{\omega} \right)^k.
\]

If the integral

\[
\int_{a}^{\infty} \left| \frac{d}{d\omega} C^k_\lambda(\omega) \right| d\omega < \infty,
\]

then \( \sum a_n \) is said to be absolutely summable by Rieszian means of order \( k \) and type \( \lambda \), or summable \( | R, \lambda, k \).

The case \( \lambda_n = n \) is of particular interest in this paper. Summability \( | R, n, k \) has been shown by J. M. Hyslop [3] to be equivalent to absolute Cesàro summability of order \( k \), or summability \( | C, k \). One of the principal results shown by Obreschkoff was the consistency of the \( | R, n, k \) means; that is, he showed that summability \( | R, n, k \) implies summability \( | R, n, k' \), where \( k' > k \).

In this paper we introduce a method of absolute summability based upon the \((\alpha, \beta)\) method of summability defined by Bosanquet and Linfoot [1]. Just as the Bosanquet-Linfoot method generalized Riesz's arithmetic mean \((R, n, \alpha)\), the method given here will generalize absolute Rieszian summability \( | R, n, \alpha \).

Definition 2. A series \( \sum a_n \) is said to be absolutely summable \((\alpha, \beta)\), or summable \( | \alpha, \beta \), where \( \alpha > 0 \) or \( \alpha = 0, \beta > 0 \), if for each sufficiently large \( C \),

\[
(1) \quad \int_{0}^{\infty} \left| \frac{d}{d\omega} A_{\alpha, \beta}(\omega) \right| d\omega < \infty,
\]

where

\[
(2) \quad A_{\alpha, \beta}(\omega) = \sum_{n < \omega} B \left( 1 - \frac{n}{\omega} \right)^{\alpha} \log^{-\beta} \frac{C}{1 - n/\omega} a_n,
\]

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and $B = \log B$. Summability $|0, 0|$ is defined to be absolute convergence.

Thus $|\alpha, 0|$ summability is the same as $|R, n, \alpha|$ summability. Condition (1) is equivalent to the bounded variation of $A_{\alpha,\beta}(\omega)$ in $(0, \infty)$. (See [2, p. 605].)

In the present paper it will be proved that $|\alpha, \beta|$ summability is consistent in the following sense: $|\alpha, \beta|$ summability implies $|\alpha', \beta'|$ summability, where either $\alpha' > \alpha$ or $\alpha' = \alpha$, $\beta' > \beta$. In a future paper, the authors propose to show some applications of $|\alpha, \beta|$ summability analogous to known results for absolute Rieszian, or Cesàro, summability.

2. Lemmas.

**Lemma 1.** Let $f(x)$, $k(u)$, and $K(u)$ satisfy the following conditions:

(i) For some $n \geq 0$, $V_0^n(x^{-n}f(x)) < \infty$ for all $T > 0$. (It will be assumed throughout that for $x = 0$, the function $x^{-n}f(x)$ is replaced by $\lim_{x \to +\infty} x^{-n}f(x)$.)

(ii) $k(u)$ is absolutely continuous in $[0, 1]$.

(iii) $K(u)$ is positive, continuously differentiable in $[0, 1]$, Lebesgue integrable over $[0, 1]$, $\lim_{u \to -1} K(u) = +\infty$ and $uK'(u)/K(u)$ is non-decreasing.

Let

$$F(x) = x^{-n} \int_0^1 K(u)f(xu)du; \quad G(x) = x^{-n} \int_0^1 k(u)K(u)f(xu)du.$$  

Then $V_0^n G(x) \leq \gamma V_0^n F(x)$, where $\gamma = \int_0^1 |k'(u)|du + |k(1)|$.

**Proof.** For $T > 0$ let $p$ be a partition, $0 = x_0 < x_1 < \cdots < x_N = T$, of $[0, T]$. Corresponding to this partition let us define

$$\Delta(f, u) = x_i^{-n} f(ux_i) - x_{i-1}^{-n} f(ux_{i-1})$$  

and

$$\Delta G_i = G(x_i) - G(x_{i-1}), \quad i = 1, 2, \cdots, N.$$

Then

$$(3) \quad \sum_{(p)} |\Delta G_i| = \sum_{(p)} \left| \int_0^1 k(u)K(u)\Delta(f, u)du \right|.$$  

An integration by parts of the right side of (3) leads to the inequality

$$(4) \quad \sum_{(p)} |\Delta G_i| \leq C_1 + \int_0^1 \left| k'(u) \right| \sum_{(p)} \left| \int_0^u K(l)\Delta(f, l)dl \right| du,$$  

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where \( C_1 = |k(1)| V_0^T F(x) \). Since \( \sum_{(p)} | \int_0^u K(t) \Delta(f_i, t) dt | \) is a continuous function of \( u \), (4) becomes, with the aid of the first mean-value theorem,

\[
\sum_{(p)} | \Delta G_i | \leq C_1 + C_2 \sum_{(p)} \left| \int_0^{u_0} K(u) \Delta(f_i, u) du \right|,
\]

where \( C_2 = \| k'(u) \| du \) and \( 0 \leq u_0 \leq 1 \).

If \( u_0 = 0 \) or 1, the right side of (5) is clearly no greater than \( \gamma V_0^T F(x) \), where \( \gamma = \int_0^1 |k'(u)| du + |k(1)|. \) If \( 0 < u_0 < 1 \), then after changing variables and integrating by parts, (5) becomes

\[
\sum_{(p)} | \Delta G_i | \leq C_1 + C_2 \left\{ \sum_{(p)} \left| \frac{u_0 K(u u_0)}{K(u)} \int_0^u K(t) \Delta(f_i, uu_0) dt \right| \right\}^{u-1}_{u=0} - u_0 \int_0^1 \frac{d}{du} \left( \frac{K(u u_0)}{K(u)} \right) \int_0^u K(t) \Delta(f_i, uu_0) dt du \right\}.
\]

But hypothesis (iii) implies that the integrated part vanishes at both limits, and that \( (d/du) \left\{ K(u u_0)/K(u) \right\} \leq 0. \) Again applying the first mean-value theorem, it follows from (6) that

\[
\sum_{(p)} | \Delta G_i | \leq C_1 + C_2 u_0 \sum_{(p)} \left| \int_0^{u_1} K(u) \Delta(f_i, uu_0) du \right|,
\]

where \( 0 \leq u_1 \leq 1 \).

Repetition of the steps leading from (5) to (7) gives the result,

\[
\sum_{(p)} | \Delta G_i | \leq C_1 + C_2 \Pi_m \sum_{(p)} \left| \int_0^1 K(u u_m) \Delta(f_i, uu_m) du \right|,
\]

where

\[
\Pi_m = \prod_{r=0}^{m} u_r, \quad 0 \leq u_r \leq 1, \quad u_r \neq 0, \ 1 \text{ for } r < m, \quad m = 1, 2, \ldots.
\]

From (8) we shall deduce that

\[
\sum_{(p)} | \Delta G_i | \leq \gamma V_0^T F(x).
\]

There are two cases to consider.

**Case 1.** For some \( m \), either \( u_m = 0 \) or 1. It is not difficult to verify then that \( \sum_{(p)} | \Delta G_i | \leq C_0 \), or \( \sum_{(p)} | \Delta G_i | \leq C_1 + C_0 (\Pi_m)^{n+1} V_0^T \Pi_m F(x) \), respectively. In either case (9) is clearly satisfied. This case for \( m = 0 \) has been settled already.
Case 2. Suppose \( u_m \neq 0, 1 \) for all \( m \). Since \( \{\Pi_m\} \) is a monotone sequence, \( \Pi_m \to L \) as \( m \to \infty \), \( 0 \leq L < 1 \). If \( L = 0 \) then

\[
\Pi_m \sum_{(p)} \left| \int_0^1 K(u_m) \Delta(f, u L_m) \, du \right| \leq 2 M N (\Pi_m)^{n+1} \int_0^1 K(u) \, du = o(1)
\]
as \( m \to \infty \), where \( M = \text{u.b.} \left[ x^{-\gamma} f(x) \right] \) over \([0, T]\). Hence (9) holds when \( L = 0 \).

Finally, if \( L \neq 0 \), then necessarily \( \lim_{m \to \infty} u_m = 1 \). Since each integrand in (8) is majorized by a summable function, a well-known theorem of Lebesgue integration may be applied to (8) to give

\[
\sum_{(p)} | \Delta G_\chi | \leq C_1 + C_2 L \sum_{(p)} \left| \int_0^1 K(u) \Delta(f, u L) \, du \right|
\]
\[
\leq C_1 + C_2 L^{n+1} V_0^T F(x)
\]
\[
\leq \gamma V_0^T F(x).
\]

Thus the truth of (9) has been established for each partition \( p \) and each \( T > 0 \). From (9) it follows that \( V_0^T G(x) \leq \gamma V_0^T F(x) \), and from this the lemma.

**Lemma 2.** Lemma 1 remains valid if condition (iii) is replaced by:

(iii)*. \( K(u) \) is constant in \([0, 1]\).

**Proof.** An argument similar to that in the preceding lemma will show that (8) also holds under (iii)*. Then (9) is easily verified, and the conclusion follows.

3. The consistency theorem.

**Theorem.** If \( \sum a_n \) is summable \( | \alpha, \beta | \), then it is summable \( | \alpha', \beta' | \), for \( \alpha' > \alpha \), or \( \alpha' = \alpha, \beta' > \beta \).

**Proof.**

Case 1. \( \alpha = \beta = 0 \). We must show that absolute convergence of the series implies \( | \alpha', \beta' | \) summability, where \( \alpha' > 0 \) or \( \alpha' = 0, \beta' > 0 \). Let

\[
\Phi_{\alpha, \beta}(u) = B u^\alpha \log^{-\beta} \frac{C}{u}, \quad \text{if} \quad u \neq 0,
\]
\[
\Phi_{\alpha, \beta}(0) = 0, \quad \text{if} \quad \alpha > 0 \quad \text{or} \quad \alpha = 0, \beta > 0.
\]

Then, what we have to show is the convergence of the integral

\[
\int_0^{\infty} \left| \frac{1}{\omega^2} \sum_{n<\omega} \Phi_{\alpha', \beta'} \left( 1 - \frac{n}{\omega} \right) n a_n \right| \, d\omega.
\]
Noting that for $n < \omega$, $\Phi_{\alpha', \beta'}(1 - n/\omega) > 0$ for sufficiently large $C$, we have

$$
\int_0^\infty \left| \frac{1}{\omega^2} \sum_{n<\omega} \Phi_{\alpha', \beta'} \left(1 - \frac{n}{\omega}\right) n a_n \right| d\omega
\leq \int_0^\infty \sum_{n<\omega} |a_n| \frac{n}{\omega^2} \Phi_{\alpha', \beta'} \left(1 - \frac{n}{\omega}\right) d\omega
\leq \sum_{n=0}^\infty |a_n| \int_n^\infty \frac{n}{\omega^2} \Phi_{\alpha', \beta'} \left(1 - \frac{n}{\omega}\right) d\omega
= \sum_{n=0}^\infty |a_n| \int_0^1 \Phi_{\alpha', \beta'}(u) du
= \sum_{n=0}^\infty |a_n| .
$$

The result now follows, since $\sum |a_n|$ is finite.

**Case 2.** $\alpha > 0$, or $\alpha = 0$, $\beta > 0$. In this case it is known [1, p. 209] that $A_{\alpha, \beta}(\omega)$ has the integral representation,

$$
A_{\alpha, \beta}(\omega) = \int_0^1 \Phi_{\alpha, \beta}(1 - u) A(\omega u) du,
$$

where $A(x) = \sum_{n \leq x} a_n$. Let $h = [\alpha]$; then as in [1, p. 216] $A_{\alpha, \beta}(\omega)$ may be written in the following forms:

$$
A_{\alpha, \beta}(\omega) = \omega^{-j} \int_0^1 \Phi_{\alpha, \beta}^{(j+1)}(1 - u) A_j(\omega u) du,
$$

for $j = 0, 1, \ldots, h$, if $\alpha = h$, $\beta > 0$ or $h < \alpha < h+1$; for $j = 0, 1, \ldots, h-1$, if $\alpha = h + 1$, $\beta \leq 0$; where $A_j(x) = \int_0^x A_{j-1}(t) dt$ and $A_0(x) = A(x)$.

By choosing the appropriate form in (11), one finds that for $\alpha' > \alpha$ or $\alpha' = \alpha$, $\beta' > \beta$,

$$
A_{\alpha', \beta'}(\omega) = \omega^{-(h-1)} \int_0^1 \Phi_{\alpha', \beta'}^{(h)}(1 - u) \Phi_{\alpha, \beta}^{(h)}(1 - u) A_{h-1}(\omega u) du
$$

when $\alpha = h$, $\beta \leq 0$, and

$$
A_{\alpha', \beta'}(\omega) = \omega^{-h} \int_0^1 \Phi_{\alpha', \beta'}^{(h+1)}(1 - u) \Phi_{\alpha, \beta}^{(h+1)}(1 - u) A_h(\omega u) du
$$

when $\alpha = h$, $\beta > 0$ or $h < \alpha < h+1$.

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1 For justification of interchange of order of summation and integration, see, e.g., Titchmarsh [6, p. 348].
A routine calculation shows that the first and second factors of the integrands (12) and (13) satisfy the requirements for $k(u)$ and $K(u)$, respectively, in Lemma 1 or 2 (whichever is applicable) for $C$ sufficiently large. The theorem now follows immediately from these two lemmas.

REFERENCES