A GENERALIZATION OF ABSOLUTE RIESZIAN SUMMABILITY

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1. Introduction. Absolute Rieszian summability was defined in 1928 by N. Obreschkoff [4; 5] as follows:

Definition 1. Let $k>0$, and $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n$, $\lambda_n \to \infty$ as $n \to \infty$. Let

$$C^k_\lambda(\omega) = \sum_{\lambda_n \leq \omega} a_n \left(1 - \frac{\lambda_n}{\omega}\right)^k.$$

If the integral

$$\int_{a}^{\infty} \left| \frac{d}{d\omega} C^k_\lambda(\omega) \right| d\omega < \infty, \quad a \geq 0,$$

then $\sum a_n$ is said to be absolutely summable by Rieszian means of order $k$ and type $\lambda$, or summable $|P, \lambda, k|$. The case $\lambda_n = n$ is of particular interest in this paper. Summability $|R, n, k|$ has been shown by J. M. Hyslop [3] to be equivalent to absolute Cesàro summability of order $k$, or summability $|C, k|$. One of the principal results shown by Obreschkoff was the consistency of the $|P, \lambda, k|$ means; that is, he showed that summability $|P, \lambda, k|$ implies summability $|P, \lambda, k'|$, where $k' > k$.

In this paper we introduce a method of absolute summability based upon the $(\alpha, \beta)$ method of summability defined by Bosanquet and Linfoot [1]. Just as the Bosanquet-Linfoot method generalized Riesz's arithmetic mean $(R, n, \alpha)$, the method given here will generalize absolute Rieszian summability $|R, n, \alpha|$. 

Definition 2. A series $\sum a_n$ is said to be absolutely summable $(\alpha, \beta)$, or summable $|\alpha, \beta|$, where $\alpha > 0$ or $\alpha = 0$, $\beta > 0$, if for each sufficiently large $C$,

$$\int_{0}^{\infty} \left| \frac{d}{d\omega} A_{\alpha, \beta}(\omega) \right| d\omega < \infty,$$

where

$$A_{\alpha, \beta}(\omega) = \sum_{n < \omega} B \left(1 - \frac{n}{\omega}\right)^\alpha \log^{-\beta} \frac{C}{1 - n/\omega} a_n,$$

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and $B = \log^\beta C$. Summability $|0, 0|$ is defined to be absolute convergence.

Thus $|\alpha, 0|$ summability is the same as $|R, n, \alpha|$ summability. Condition (1) is equivalent to the bounded variation of $A_{\alpha, \beta}(\omega)$ in $(0, \infty)$. (See [2, p. 605].)

In the present paper it will be proved that $|\alpha, \beta|$ summability is consistent in the following sense: $|\alpha, \beta|$ summability implies $|\alpha', \beta'|$ summability, where either $\alpha' > \alpha$ or $\alpha' = \alpha$, $\beta' > \beta$. In a future paper, the authors propose to show some applications of $|\alpha, \beta|$ summability analogous to known results for absolute Rieszian, or Cesàro, summability.

2. Lemmas.

**Lemma 1.** Let $f(x)$, $k(u)$, and $K(u)$ satisfy the following conditions:

(i) For some $n \geq 0$, $V_0^n(x^{-nf(x)}) < \infty$ for all $T > 0$. (It will be assumed throughout that for $x = 0$, the function $x^{-nf(x)}$ is replaced by $\lim_{x \to 0^+} x^{-nf(x)}$.)

(ii) $k(u)$ is absolutely continuous in $[0, 1]$.

(iii) $K(u)$ is positive, continuously differentiable in $[0, 1]$, Lebesgue integrable over $[0, 1]$, $\lim_{u \to 1^-} K(u) = +\infty$ and $uK'(u)/K(u)$ is non-decreasing.

Let

$$F(x) = x^{-n} \int_0^1 K(u)f(xu)du; \quad G(x) = x^{-n} \int_0^1 k(u)K(u)f(xu)du.$$ 

Then $V_0^n G(x) \leq \gamma V_0^n F(x)$, where $\gamma = \int_0^1 |k'(u)| |K(u)| du + |k(1)|$.

**Proof.** For $T > 0$ let $\rho$ be a partition, $0 = x_0 < x_1 < \cdots < x_N = T$, of $[0, T]$. Corresponding to this partition let us define

$$\Delta(f_i, u) = x_i^{-n} f(ux_i) - x_{i-1}^{-n} f(ux_{i-1})$$

and

$$\Delta G_i = G(x_i) - G(x_{i-1}), \quad i = 1, 2, \cdots, N.$$ 

Then

$$\sum_{(p)} |\Delta G_i| = \sum_{(p)} \left| \int_0^1 k(u)K(u)\Delta(f_i, u)du \right|. \quad (3)$$

An integration by parts of the right side of (3) leads to the inequality

$$\sum_{(p)} |\Delta G_i| \leq C_1 + \int_0^1 |k'(u)| \left| \sum_{(p)} \int_0^\infty K(t)\Delta(f_i, t)dt \right| du, \quad (4)$$
where \( C_1 = |k(1)| V_0^T F(x) \). Since \( \sum_{(p)} |\int_0^u K(t) \Delta(f_i, t) dt| \) is a continuous function of \( u \), (4) becomes, with the aid of the first mean-value theorem,

\[
\sum_{(p)} |\Delta G_i| \leq C_1 + C_2 \sum_{(p)} \left| \int_0^{u_0} K(u) \Delta(f_i, u) du \right|,
\]

where \( C_2 = \int_0^1 k'(u) \, du \) and \( 0 \leq u_0 \leq 1 \).

If \( u_0 = 0 \) or \( 1 \), the right side of (5) is clearly no greater than \( \gamma V_0^T F(x) \), where \( \gamma = \int_0^1 |k'(u)| \, du + |k(1)| \). If \( 0 < u_0 < 1 \), then after changing variables and integrating by parts, (5) becomes

\[
\sum_{(p)} |\Delta G_i| \leq C_1 + C_2 \left\{ \sum_{(p)} \left| \frac{u_0 K(uu_0)}{K(u)} \int_0^u K(t) \Delta(f_i, tu_0) dt \right| \right\}^{u-1} - \left. u_0 \int_0^1 \frac{d}{du} \left( \frac{K(uu_0)}{K(u)} \right) \int_0^u K(t) \Delta(f_i, tu_0) dt \right| du \}
\]

But hypothesis (iii) implies that the integrated part vanishes at both limits, and that \( (d/du) \{ K(uu_0)/K(u) \} \leq 0 \). Again applying the first mean-value theorem, it follows from (6) that

\[
\sum_{(p)} |\Delta G_i| \leq C_1 + C_2 u_0 \sum_{(p)} \left| \int_0^{u_1} K(u) \Delta(f_i, uu_0) du \right|,
\]

where \( 0 \leq u_1 \leq 1 \).

Repetition of the steps leading from (5) to (7) gives the result,

\[
\sum_{(p)} |\Delta G_i| \leq C_1 + C_2 \Pi_m \sum_{(p)} \left| \int_0^1 K(uu_m) \Delta(f_i, uu_m) du \right|,
\]

where

\[
\Pi_m = \prod_{\nu=0}^{m} u_\nu, \quad 0 \leq u_\nu \leq 1, \quad u_\nu \neq 0, 1 \text{ for } \nu < m, \quad m = 1, 2, \ldots
\]

From (8) we shall deduce that

\[
\sum_{(p)} |\Delta G_i| \leq \gamma V_0^T F(x).
\]

There are two cases to consider.

Case 1. For some \( m \), either \( u_m = 0 \) or \( 1 \). It is not difficult to verify then that \( \sum_{(p)} |\Delta G_i| \leq C_0 \), or \( \sum_{(p)} |\Delta G_i| \leq C_1 + C_0 (\Pi_m)^{n+1} V_0^T \Pi_m F(x) \), respectively. In either case (9) is clearly satisfied. This case for \( m = 0 \) has been settled already.
Case 2. Suppose \( u_m \neq 0, 1 \) for all \( m \). Since \( \{\Pi_m\} \) is a monotone sequence, \( \Pi_m \rightarrow L \) as \( m \rightarrow \infty, 0 \leq L < 1 \). If \( L = 0 \) then

\[
\Pi_m \sum_{(p)} \int_0^1 K(uu_m) \Delta(f, u\Pi_m) du \leq 2MN(\Pi_m)^{n+1} \int_0^1 K(u) du = o(1)
\]
as \( m \rightarrow \infty \), where \( M = \text{u.b.} \ x^{-\gamma}(x) \text{ over } [0, T] \). Hence (9) holds when \( L = 0 \).

Finally, if \( L \neq 0 \), then necessarily \( \lim_{m \to \infty} u_m = 1 \). Since each integrand in (8) is majorized by a summable function, a well-known theorem of Lebesgue integration may be applied to (8) to give

\[
\sum_{(p)} |\Delta G_i| \leq C_1 + C_2L \sum_{(p)} \int_0^1 K(u) \Delta(f, uL) du
\]

Thus the truth of (9) has been established for each partition \( p \) and each \( T > 0 \). From (9) it follows that \( V_0^T G(x) \leq V_0^T F(x) \), and from this the lemma.

Lemma 2. Lemma 1 remains valid if condition (iii) is replaced by:

(iii)*. \( K(u) \) is constant in \([0, 1]\).

Proof. An argument similar to that in the preceding lemma will show that (8) also holds under (iii)*. Then (9) is easily verified, and the conclusion follows.

3. The consistency theorem.

Theorem. If \( \sum a_n \) is summable \( |\alpha, \beta| \), then it is summable \( |\alpha', \beta'| \), for \( \alpha' > \alpha \), or \( \alpha' = \alpha, \beta' > \beta \).

Proof. Case 1. \( \alpha = \beta = 0 \). We must show that absolute convergence of the series implies \( |\alpha', \beta'| \) summability, where \( \alpha' > 0 \) or \( \alpha' = 0, \beta' > 0 \). Let

\[
\Phi_{\alpha,\beta}(u) = Bu^{\alpha} \log^{-\beta} \frac{C}{u}, \quad \text{if } u \neq 0,
\]

\[
\Phi_{\alpha,\beta}(0) = 0, \quad \text{if } \alpha > 0 \quad \text{or} \quad \alpha = 0, \beta > 0.
\]

Then, what we have to show is the convergence of the integral

\[
\int_0^\infty \left| \frac{1}{\omega^2} \sum_{n<\omega} \Phi_{\alpha',\beta'} \left(1 - \frac{n}{\omega}\right) na_n \right| d\omega.
\]
Noting that for \( n < \omega, \Phi'_{\alpha', \theta'}(1 - n/\omega) > 0 \) for sufficiently large \( C \), we have\(^1\)

\[
\int_0^\infty \left| \frac{1}{\omega^2} \sum_{n<\omega} \Phi'_{\alpha', \theta'} \left( 1 - \frac{n}{\omega} \right) \right| d\omega \\
\leq \int_0^\infty \sum_{n<\omega} \left| a_n \right| \frac{n}{\omega^2} \Phi'_{\alpha', \theta'} \left( 1 - \frac{n}{\omega} \right) d\omega \\
\leq \sum_{n=0}^\infty \left| a_n \right| \int_n^{\infty} \frac{n}{\omega^2} \Phi'_{\alpha', \theta'} \left( 1 - \frac{n}{\omega} \right) d\omega \\
= \sum_{n=0}^\infty \left| a_n \right| \int_0^1 \Phi'_{\alpha', \theta'}(u)du \\
= \sum_{n=0}^\infty \left| a_n \right|
\]

The result now follows, since \( \Sigma |a_n| \) is finite.

Case 2. \( \alpha > 0 \), or \( \alpha = 0, \beta > 0 \). In this case it is known \[1, p. 209\] that \( A_{\alpha, \beta}(\omega) \) has the integral representation,

\[
A_{\alpha, \beta}(\omega) = \int_0^{1} \Phi'_{\alpha, \beta}(1 - u) A(\omega u)du,
\]

where \( A(x) = \sum_{n \leq x} a_n \). Let \( h = \lfloor \alpha \rfloor \); then as in \[1, p. 216\] \( A_{\alpha, \beta}(\omega) \) may be written in the following forms:

\[
(11) A_{\alpha, \beta}(\omega) = \omega^j \int_0^{1} \Phi_{\alpha, \beta}^{(j+1)}(1 - u) A_j(\omega u)du,
\]

for \( j = 0, 1, \ldots, h \), if \( \alpha = h, \beta > 0 \) or \( h < \alpha < h + 1 \); for \( j = 0, 1, \ldots, h - 1 \), if \( \alpha = h \geq 1, \beta \leq 0 \); where \( A_j(x) = \int_0^x A_{j-1}(t)dt \) and \( A_0(x) = A(x) \).

By choosing the appropriate form in (11), one finds that for \( \alpha' > \alpha \) or \( \alpha' = \alpha, \beta' > \beta \),

\[
(12) A_{\alpha', \beta'}(\omega) = \omega^{-(h-1)} \int_0^{1} \frac{\Phi_{\alpha', \beta'}^{(h)}(1 - u)}{\Phi_{\alpha, \beta}^{(h)}(1 - u)} \Phi_{\alpha, \beta}^{(h)}(1 - u) A_{h-1}(\omega u)du
\]

when \( \alpha = h, \beta \leq 0, \) and

\[
(13) A_{\alpha', \beta'}(\omega) = \omega^{-h} \int_0^{1} \frac{\Phi_{\alpha', \beta'}^{(h+1)}(1 - u)}{\Phi_{\alpha, \beta}^{(h+1)}(1 - u)} \Phi_{\alpha, \beta}^{(h+1)}(1 - u) A_{h}(\omega u)du
\]

when \( \alpha = h, \beta > 0 \) or \( h < \alpha < h + 1 \).

\(^1\) For justification of interchange of order of summation and integration, see, e.g., Titchmarsh \[6, p. 348\].
A routine calculation shows that the first and second factors of the integrands (12) and (13) satisfy the requirements for $k(u)$ and $K(u)$, respectively, in Lemma 1 or 2 (whichever is applicable) for $C$ sufficiently large. The theorem now follows immediately from these two lemmas.

REFERENCES


Florida State University and
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