EQUI-ABSOLUTE CONVERGENCE OF EIGENFUNCTION EXPANSIONS[1]

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1. **Introduction.** The eigenfunctions we allow are described in §2, and examples are given in §4. We prove in §3 that a function with small enough support has an absolutely convergent eigenfunction expansion if and only if its Fourier series converges absolutely. In §5 we mention similar results for integral transforms.

We first point out two examples which show that our results are not completely trivial. P. Lévy [**5**, p. 5] has shown that for

$$f(x) = \left[\log\left(\frac{\sin x}{2}\right)\right]^{-1}, \qquad 0 < x < \pi,$$

the cosine series converges absolutely while the sine series does not. Also, the cosine series of

$$g(x) = \sum_{1}^{\infty} k^{-2} \sin 2^k x, \qquad 0 < x < \pi,$$

is not absolutely convergent, while the sine series obviously is (cf. [**1**, p. 632]). We shall see in Theorem 4 that this is due to a boundary effect.

The results of this paper are a generalization of the following theorem of N. Wiener [**8**, p. 14].

THEOREM 1. *Let f be continuous on $(0, \infty)$ and let $f(x) = 0$ for x outside $[c, d]$. Let $[c, d] \subset (a, b) \subset (0, \infty)$, b finite. Then the Fourier cosine transform of f is in $L_1(0, \infty)$ if and only if the Fourier series of f on (a, b) converges absolutely.*

2. **Conditions on the orthogonal functions.** For an interval (a, b), either finite or infinite, let $\{v_k\}$ be a complete orthogonal set of functions in real $L_2(a, b)$. Let there be numbers θ_k and λ_k such that uniformly on every $[c, d] \subset (a, b)$

$$v_k(x) = \cos(\lambda_k x - \theta_k) + o(1) \tag{1}$$

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as $k \to \infty$. Here we assume that

$$0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots, \qquad \lambda_k \to \infty. \tag{2}$$

Besides this, let there be a number η such that for every μ

$$\sum_{\mu \leqq \lambda_k < \mu+1} \|v_k\|^{-2} \leqq \eta. \tag{3}$$

More specifically, for each $[c, d] \subset (a, b)$ let

$$v_k(x) = \phi_k(x) \cos \lambda_k x + \psi_k(x) \sin \lambda_k x. \tag{4}$$

Here the ϕ_k and ψ_k are assumed to be absolutely continuous. Also we assume that on $[c, d]$ uniformly in k the ϕ_k, ϕ_k', ψ_k, ψ_k' are bounded and the ϕ_k', ψ_k' are of bounded variation.

Remark. For many orthogonal systems it is more convenient to note that (3) is a consequence of the requirement that the number of λ_k in each interval of unit length be uniformly bounded. We have from (1)

$$\int_a^b v_k^2 \geqq \int_c^d v_k^2 \sim \int_c^d \cos^2(\lambda_k x - \theta_k) dx \sim (d - c)/2.$$

Thus there is a positive σ such that

$$\sum_{\mu \leqq \lambda_k < \mu+1} \|v_k\|^{-2} \leqq \sigma \sum_{\mu \leqq \lambda_k \leqq \mu+1} 1,$$

from which (3) follows.

3. **Absolute convergence of orthogonal expansions.** We first mention two theorems which allow us to confine our attention to the absolute convergence of the coefficients.

Theorem 2. *Let the v_k satisfy the conditions of the previous section. Let $\sum |c_k| < \infty$. Then $\sum |c_k v_k(x)| < \infty$, $a < x < b$, and $\sum c_k v_k(x)$ converges uniformly on every $[c, d] \subset (a, b)$.*

This follows immediately from (1).

In the opposite direction we have

Theorem 3. *Let the v_k satisfy the conditions of §2. If $\sum |c_k v_k(x)| < \infty$ on a set of positive measure, then $\sum |c_k| < \infty$.*

Let E be the set of absolute convergence. Since we have

$$\liminf_{k \to \infty} \int_E |v_k(x)| dx > 0$$

from (1), the result follows from Lusin's theorem [**3**, p. 153].

LEMMA 1. *Suppose the conditions of* §2 *are satisfied. Let*

$$f(x) = g(x) \cos \mu x \tag{5}$$

or

$$f(x) = g(x) \sin \mu x \tag{6}$$

where $\mu \geqq 0$, g *is absolutely continuous on* (a, b), $g(x)=0$ *outside an interval* $[c, d] \subset (a, b)$, *and* g' *is of bounded variation on* $[c, d]$. *Let*

$$B_k = (f, v_k)\|v_k\|^{-2}. \tag{7}$$

Then there is a constant M, *independent of* μ *and the behavior of* f *on* $[c, d]$, *such that*

$$\sum |B_k| \leqq M \operatorname*{var}_{[c,d]} g'. \tag{8}$$

It is sufficient to consider only (5) since the other case is similar. By equation (4)

$$B_k = \|v_k\|^{-2}\left\{\int_a^b g(x)\phi_k(x) \cos \mu x \cos \lambda_k x dx \right.$$

$$\left. + \int_a^b g(x)\psi_k(x) \cos \mu x \sin \lambda_k x dx\right\}.$$

Since the second integral in curly brackets is similar to the first, we consider only the first.

Since $g(x)=0$ for x outside $[c, d]$, we estimate

$$C_k = \|v_k\|^{-2}\int_c^d g(x)\phi_k(x) \cos \mu x \cos \lambda_k x dx. \tag{9}$$

A computation involving integration by parts twice shows that

$$|C_k| \leqq \|v_k\|^{-2} \min\left[(d-c) \sup_{[c,d]} |g\phi_k|, 2(\lambda_k-\mu)^{-2} \operatorname*{var}_{[c,d]} (g\phi_k)'\right].$$

However,

$$\operatorname{var} (g\phi_k)' \leqq \sup |\phi_k| \operatorname{var} g' + 2(d-c) \sup |\phi_k'| \sup |g'|$$

$$+ \operatorname{var} \phi_k' \sup |g|.$$

But $\sup |g| \leqq (d-c) \sup |g'|$, and $\sup |g'| \leqq \operatorname{var} g'$ since the average value of g' is zero. Thus from the assumptions on the ϕ_k,

$$|C_k| \leqq M_0\|v_k\|^{-2} \min[1, (\lambda_k-\mu)^{-2}] \operatorname{var} g'. \tag{10}$$

We now estimate the sum using (3)

$$\sum |C_k| \leqq M_0 \left\{ \sum_{\mu-1 \leqq \lambda_k < \mu+1} \|v_k\|^{-2} \right.$$

$$\left. + \sum_{j=-\infty; j \neq -1,0}^{\infty} \sum_{\mu+j \leqq \lambda_k < \mu+j+1} (\lambda_k - \mu)^{-2} \|v_k\|^{-2} \right\} \operatorname{var} g'$$

$$\leqq M_0 \left\{ 2\eta + 2\eta \sum_{1}^{\infty} j^{-2} \right\} \operatorname{var} g'.$$

We are ready for the main theorem of the paper once the trapezoid function is defined. Let $\alpha < \gamma < \delta < \beta$. Then $T(x) = 1$ on $[\gamma, \delta]$; $T(x) = 0$ outside (α, β); and $T(x)$ is linear on $[\alpha, \gamma]$ and on $[\delta, \beta]$.

THEOREM 4. *Let the functions v_k be orthogonal on (a, b) and the functions w_j orthogonal on (a', b'), both systems satisfying the conditions in §2. Let T be a trapezoid function with support*

$$[\alpha, \beta] \subset (a, b) \cap (a', b').$$

Let $f = \sum c_k v_k$ with $\sum |c_k| < \infty$. Then $Tf = \sum d_j w_j$ with $\sum |d_j| < \infty$.

We know from Theorem 2 that the series $\sum c_k v_k$ converges uniformly on $[\alpha, \beta]$. Hence,

$$d_j = \sum_k c_k \|w_j\|^{-2} \int_\alpha^\beta T v_k w_j$$

$$= \sum_k c_k \|w_j\|^{-2} \left\{ \int_\alpha^\beta T(x) \phi_k(x) \cos \lambda_k x w_j(x) dx \right.$$

$$\left. + \int_\alpha^\beta T(x) \psi_k(x) \sin \lambda_k x w_j(x) dx \right\}.$$

By Lemma 1 and the conditions on the ϕ_k and ψ_k

$$\sum_k |d_j| \leqq \sum_k |c_k| M \{ \operatorname{var} (T\phi_k)' + \operatorname{var} (T\psi_k)' \} < \infty.$$

In this theorem we may use orthogonal trigonometric functions as either the v_k or the w_j. Hence we have the

COROLLARY. *Let the functions v_k satisfy the conditions of §2. Let $f(x) = 0$ outside $[\alpha, \beta]$. Let*

$$[\alpha, \beta] \subset (a', b') \subset (a, b),$$

a', b' finite. Then the v_k expansion of f on (a, b) converges absolutely if and only if the Fourier expansion of f on (a', b') converges absolutely.

4. **Examples.** We shall show that the common Sturm-Liouville eigenfunctions satisfy our conditions. Let $\{v_k, \lambda_k\}$ be the solutions of the problem

$$y'' + \lambda^2 y = ry, \qquad -\infty \leqq a < x < b \leqq \infty, \tag{11}$$

with some homogeneous boundary conditions. We assume that r is real valued and that the eigenvalues are discrete, with $\lambda_k \to \infty$. By adding a constant to r we can obtain

$$1 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots. \tag{12}$$

LEMMA 2. *Let r be of bounded variation on every $[c, d] \subset (a, b)$. Then with the proper normalization we have uniformly on $[c, d]$*

$$v_k(x) = \cos(\lambda_k x - \theta_k) + o(1) \tag{1}$$

as $k \to \infty$. Also

$$v_k(x) = \phi_k(x) \cos \lambda_k x + \psi_k(x) \sin \lambda_k x \tag{4}$$

with the ϕ_k and ψ_k absolutely continuous. The ϕ_k, ϕ_k', ψ_k, ψ_k' are bounded and the ϕ_k', ψ_k' are of bounded variation uniformly in k on $[c, d]$.

By the variation of constants formula there are constants A_k, θ_k such that

$$v_k(x) = A_k \cos(\lambda_k x - \theta_k) + \lambda_k^{-1} \int_c^x \sin \lambda_k(x - t) r(t) v_k(t) dt. \tag{13}$$

We normalize so that $A_k = 1$. The estimate (1) then follows easily.

To prove the rest of the lemma, we see from (13) that we should take

$$\phi_k(x) = \cos \theta_k - \lambda_k^{-1} \int_c^x \sin \lambda_k t r(t) v_k(t) dt;$$
$$\phi_k'(x) = -\lambda_k^{-1} \sin \lambda_k x r(x) v_k(x).$$

The ϕ_k and ϕ_k' are clearly bounded on $[c, d]$ uniformly in k. By differentiating (13) we see that the functions $\lambda_k^{-1} v_k'$ are uniformly bounded. Hence, the ϕ_k' are of bounded variation on $[c, d]$ uniformly in k. The ψ_k are similar:

$$\psi_k(x) = \sin \theta_k + \lambda_k^{-1} \int_c^x \cos \lambda_k t r(t) v_k(t) dt.$$

For these eigenfunctions all the conditions of §2 will be satisfied as soon as (3) is satisfied. This is true for the following classical cases.

For most of them it is sufficient to note that the number of λ_k in each interval of unit length is bounded by a constant independent of the interval.

The eigenfunctions of the regular Sturm-Liouville problem are admissible [4, pp. 213–214, 262]

$$y'' + \lambda^2 y = ry, \qquad -\infty < a < x < b < \infty,$$

$$\sin \alpha y(a) - \cos \alpha y'(a) = 0,$$

$$\sin \beta y(b) - \cos \beta y'(b) = 0.$$

Here r is assumed to be integrable over (a, b) and of bounded variation on every $[c, d] \subset (a, b)$.

The Bessel functions

$$v_k(x) = (\lambda_k x)^{1/2} J_\mu(\lambda_k x),$$

$0 < x \leqq 1$, $\mu > -1/2$, with $J_\mu(\lambda_k) = 0$ satisfy the conditions of §2 [7, pp. 199, 506].

Another example is given by the Jacobi polynomials [6, pp. 66, 163]:

$$v_k(x) = (k+1)^{1/2} \left(\sin \frac{x}{2}\right)^{\alpha+1/2} \left(\cos \frac{x}{2}\right)^{\beta+1/2} P_k^{\alpha,\beta} (\cos x),$$

$$0 < x < \pi, \alpha > -1, \beta > -1.$$

For Laguerre and Hermite polynomials it can be shown that (3) is true. The proper form for the Laguerre functions is [6, pp. 96, 193]:

$$v_k(x) = (k+1)^{(-2\alpha+1)/4} e^{-x^2/2} x^{\alpha+1/2} L_k^{(\alpha)}(x^2), \qquad 0 < x < \infty, \alpha > -1.$$

Our normalization for the Hermite functions is [6, p. 194]:

$$v_k(x) = (k+1)^{1/4} \pi^{1/4} (k!)^{-1/2} 2^{-(k+1)/2} H_k(x) e^{-x^2/2}.$$

5. **Absolute convergence of integral transforms.** The results of §§3 and 4 can be generalized to corresponding theorems about integral transforms derived from Sturm-Liouville problems. We shall state only the results.

We impose conditions similar to those of §2. Let a real function $v(x, \lambda)$, $0 < x < \infty$, $0 < \lambda < \infty$, satisfy

$$v(x, \lambda) = \cos(\lambda x - \theta(\lambda)) + o(1) \tag{14}$$

as $\lambda \to \infty$, uniformly in x on every $[c, d] \subset (0, \infty)$. As before we require that for each $[c, d] \subset (0, \infty)$

$$v(x, \lambda) = \phi(x, \lambda) \cos \lambda x + \psi(x, \lambda) \sin \lambda x, \tag{15}$$

$c \leqq x \leqq d$, $0 < \lambda < \infty$. Here ϕ and ψ are assumed to be absolutely continuous in x for each λ. Also on $[c, d]$ uniformly in λ we require ϕ, ϕ_x, ψ, ψ_x to be bounded and ϕ_x, ψ_x to be of bounded variation.

We assume the existence of a nondecreasing function σ such that for each f in real L_2 there is a transform h:

$$\lim_{b\to\infty} \int_0^\infty \left[h(\lambda) - \int_0^b f(x)v(x, \lambda)dx\right]^2 d\sigma(\lambda) = 0, \tag{16}$$

and

$$\lim_{b\to\infty} \int_0^\infty \left[f(x) - \int_0^b h(\lambda)v(x, \lambda)d\sigma(\lambda)\right]^2 dx = 0. \tag{17}$$

Also we assume that there is a constant η independent of λ such that

$$\sigma(\lambda + 1) - \sigma(\lambda) \leqq \eta. \tag{18}$$

Except possibly for the realness of λ and (18) these conditions are satisfied by the solutions of the boundary value problem:

$$\begin{gathered} y'' + \lambda^2 y = ry, \qquad 0 < x < \infty, \\ \sin \alpha y(0) - \cos \alpha y'(0) = 0 \end{gathered} \tag{19}$$

[2, p. 232]. Here r is to be real valued and of bounded variation on every $[c, d] \subset (0, \infty)$. We assume that we have the limit point case since the limit circle case yields series expansions.

THEOREM 5. *Let $v(x, \lambda)$ satisfy the conditions listed above. Let $f \in L_2(0, \infty)$ and let $f(x) = 0$ outside $[c, d] \subset (0, \infty)$. Suppose f is represented by the transforms*

$$\begin{gathered} f(x) \sim \int_0^\infty g(t) \cos txdt, \\ f(x) \sim \int_0^\infty h(\lambda)v(x, \lambda)d\sigma(\lambda). \end{gathered} \tag{20}$$

Then $\int_0^\infty |g(t)| dt < \infty$ if and only if $\int_0^\infty |h(\lambda)| d\sigma(\lambda) < \infty$.

The proof is similar to that in §3.

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A COMPARISON THEOREM FOR SPACES OF ENTIRE FUNCTIONS

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A Hilbert space, whose elements are entire functions, is of particular interest if it has these properties:

(H1) Whenever $F(z)$ is in the space and has a nonreal zero w, the function $F(z)(z-\bar{w})/(z-w)$ is in the space and has the same norm as $F(z)$.

(H2) For every nonreal number w, the linear functional defined on the space by $F(z)\to F(w)$ is continuous.

(H3) Whenever $F(z)$ is in the space, the function $F^*(z)=\bar{F}(\bar{z})$ is in the space and has the same norm as $F(z)$. If $E(z)$ is an entire function which satisfies the inequality

$$|E(\bar{z})| < |E(z)| \tag{1}$$

for $y>0$ $(z=x+iy)$, let $E(z)=A(z)-iB(z)$, where $A(z)$ and $B(z)$ are entire functions which are real for real z, and

$$K(w,z) = [B(z)\bar{A}(w) - A(z)\bar{B}(w)]/[\pi(z-\bar{w})].$$

Let $\mathcal{H}(E)$ be the corresponding set of entire functions $F(z)$ such that

$$\|F\|^2 = \int |F(t)/E(t)|^2 dt < \infty,$$

with integration on the real axis, and

$$|F(z)|^2 \leq \|F\|^2 K(z,z)$$

for all complex z. Then, $\mathcal{H}(E)$ is a Hilbert space of entire functions

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