

ANOTHER CLASS OF INVERTIBLE OPERATORS WITHOUT SQUARE ROOTS

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1. **Introduction.** In [2], Halmos, Lumer, and Schäffer exhibited a class of invertible operators on Hilbert space possessing no (bounded) operator square roots, thereby settling what was until then an open question. This class of examples was obtained by considering the Hilbert space of all analytic functions which are square-integrable (with respect to planar Lebesgue measure) on a bounded domain D in the complex plane. In this setting they show that the "analytic position operator" defined by $(Af)(t) = tf(t)$ has a square root if and only if D contains no Jordan curve surrounding zero.

The present authors, while trying to discover the limitations of Corollary 4.2 [1], were led to a different, matrix-theoretic example of an invertible operator on Hilbert space which has no (bounded) operator square root. In fact considerations of spectra show that the class of examples exhibited here is disjoint from the class appearing in [2].

2. **An example.** The following considerations lead to the construction of the above mentioned example. Let $N = \{n_k\}_{k=1}^{\infty}$ be any strictly increasing sequence of positive integers such that $n_1 \geq 3$. For simplicity of notation in what follows, we shall write $n \in N$ to mean "for some positive integer k , $n = n_k$ " and $\{a_n\}_{n \in N}$ or simply $\{a_n\}$ instead of $\{a_{n_k}\}_{k=1}^{\infty}$. Let \mathcal{H}_n be n -dimensional Hilbert space, and let M_n be the algebra of all operators on \mathcal{H}_n . Denote by \mathcal{H} the Hilbert space $\sum_{n \in N} \oplus \mathcal{H}_n$, and by R the W^* -algebra $\sum_{n \in N} \oplus M_n$. Then of course R consists of all operators $A = \sum \oplus A_n$ where $A_n \in M_n$ and the sequence $\{\|A_n\|\}$ is bounded. The operators which we shall consider can most easily be described by setting forth matrices, so we assume given for each $n \in N$ an orthonormal basis X_n of \mathcal{H}_n , and the matrices exhibited are to be regarded as the corresponding operators. We now define the class $\Gamma \subset R$ of operators as follows: $T = \sum \oplus T_n \in \Gamma$ if and only if for each $n \in N$, T_n has the form

$$(*) \quad \begin{pmatrix} \omega_n^1 & s_n & & f_n \\ & \omega_n^2 & & \circ \\ & & \ddots & \vdots \\ \circ & & & s_n \\ & & & & \omega_n^n \end{pmatrix},$$

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where

- (1) for $1 \leq r \leq n$, $\omega_n^r = e^{i2\pi r/n}$,
- (2) the sequence $\{s_n\}_{n \in \mathbb{N}}$ satisfies $0 \leq |s_n| < \delta$ for some $\delta < 1$ and $\lim_{n \in \mathbb{N}} n|s_n| = \infty$.
- (3) the sequence $\{|f_n|\}$ is bounded.

The inverse of each T_n satisfying (*) can be computed directly, and an easy calculation shows that $\|T_n^{-1}\| \leq (1 - |s_n|)^{-1} + |f_n|$. It follows from (2) and (3) that each $T \in \Gamma$ is invertible in R , and the following lemmas enable us to determine a subclass of Γ consisting of operators having no square roots in R .

LEMMA 2.1. *Suppose that $n > 1$ and that for $1 \leq r \leq n$, λ_n^r are complex numbers satisfying $(\lambda_n^r)^2 = \omega_n^r$. Then*

$$(**) \quad \lambda_n^r = \pm e^{i\pi r/n}$$

and either

- (α) for some r satisfying $1 \leq r \leq n - 1$, $|\lambda_n^r + \lambda_n^{r+1}| < \pi/n$, or
- (β) the sign in (**) is the same for $1 \leq r \leq n$, in which case $|\lambda_n^1 + \lambda_n^n| < \pi/n$.

The proof is trivial and is omitted.

LEMMA 2.2. *Suppose that K is an operator on a Hilbert space \mathfrak{H} having the distinct eigenvalues $\{\lambda_i\}_{i \in I}$, and suppose that for $i \in I$, the eigenspace corresponding to λ_i is spanned by the vector x_i . If J is any operator on \mathfrak{H} such that $J^2 = K$, and \mathfrak{L} is any subspace of \mathfrak{H} spanned by some subset of the $\{x_i\}_{i \in I}$, then \mathfrak{L} is an invariant subspace for J .*

PROOF. It suffices to show that for $i \in I$, $Jx_i = \alpha_i x_i$ for some scalar α_i . If $Jx_i = y_i$, then $Ky_i = KJx_i = JKx_i = \lambda_i y_i$, so that by hypothesis $y_i = \alpha_i x_i$.

We obtain as immediate corollaries

COROLLARY 2.3. *If K is an $n \times n$ complex matrix in upper triangular form and having n distinct eigenvalues, and J is any square root of K , then J is also in upper triangular form.*

COROLLARY 2.4. *Suppose $K \in R$ satisfies the hypotheses of Lemma 2.2 and the additional hypothesis that the vectors $\{x_i\}_{i \in I}$ span \mathfrak{H} . Then every (bounded) square root of K on \mathfrak{H} is in R .*

For each sequence $\sigma = \{s_n\}_{n \in \mathbb{N}}$ satisfying (2) of (*), consider the operator $P_\sigma = \sum \oplus P_n \in \Gamma$ defined by taking the sequence $\{f_n\}$ to be identically zero. An induction argument similar to the one carried

3. We are now in a position to prove the

THEOREM. *There is an invertible operator B on any Hilbert space \mathfrak{H} such that B has no square root in the algebra of all bounded operators on \mathfrak{H} .*

PROOF. It is easy to see (in view of Lemma 2.2) that it suffices to prove the theorem in the case that \mathfrak{H} is separable. Let $A_\sigma = \sum \oplus A_n$ be as in Lemma 2.5, and define the operator $B \in R$ by $B = \sum \oplus (1+1/n)A_n$. Since A_σ is an invertible operator on \mathfrak{H} , so is B , and by virtue of Corollary 2.4, any (bounded) square root of B must be an operator in R . But B cannot have a square root in R , since A_σ does not, and thus the proof is complete.

As another easy corollary of Lemma 2.5, we obtain

COROLLARY 3.1. *If S is any finite W^* -algebra of type I possessing n_k -homogeneous summands with $n_k \rightarrow \infty$, then there is an invertible element $A \in S$ such that A has no square root in S .*

PROOF. Any W^* -algebra S which satisfies the hypothesis of the theorem can be written as a direct sum $S = \sum_{n \in N} \oplus S_n$ where N is an unbounded set of positive integers and each S_n is an n -homogeneous W^* -algebra [4]. If \mathfrak{X}_n is taken to be the maximal ideal space of the center of S_n , then (see [5]) S is C^* -isomorphic to the algebra S_0 which is the C^* -sum of the homogeneous W^* -algebras $\{M_n(\mathfrak{X}_n)\}_{n \in N}$ where the $M_n(\mathfrak{X}_n)$ are as defined in [1]. Thus it suffices to prove the theorem in S_0 , and an invertible element $G = \{G_n\} \in S_0$ having no square root in S_0 can be obtained by choosing $A_\sigma = \sum \oplus A_n$ as in Lemma 2.5 and then defining $G_n(t) \equiv A_n$ for $t \in \mathfrak{X}_n$ and $n \in N$. If G had a square root $J = \{J_n\} \in S_0$, then choosing arbitrary points $t_n \in \mathfrak{X}_n$ for $n \in N$, one would have $[J_n(t_n)]^2 = G_n(t_n) = A_n$, contradicting the fact that A_σ has no square root in R . Thus the proof is complete.

4. Remarks.

(1) Hille and Phillips have shown ([3], Theorem 9.5.4) that a sufficient condition for an operator A to have roots of all orders in the Banach algebra generated by A and the identity is that zero belong to the unbounded component of the resolvent set of A .

(2) Each analytic position operator appearing in [2] is such that the compression spectrum of the operator (in the terminology of [2]) contains an open set. On the other hand, it is clear that the operator B without square root exhibited above has discrete compression spectrum, so that B cannot be an analytic position operator.

(3) If N contains only odd integers, it is not hard to see that the operator B appearing in the proof of the Theorem is such that B^2 has no bounded fourth root. This shows that invertible operators can have roots of some order without having roots of all orders.

REFERENCES

1. D. Deckard and C. Pearcy, *On matrices over the ring of continuous complex-valued functions on a Stonian space*, Proc. Amer. Math. Soc. (to appear).
2. P. R. Halmos, G. Lumer, and J. Schäffer, *Square roots of operators*, Proc. Amer. Math. Soc. 4 (1953), 142-149.
3. E. Hille and R. S. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Colloq. Publ. Vol. 31, Amer. Math. Soc., Providence, R. I., 1957.
4. I. Kaplansky, *Algebras of type I*, Ann. of Math. (2) 56 (1952), 460-472.
5. C. Pearcy, *A complete set of unitary invariants for operators generating finite W^* -algebras of type I*, Pacific J. Math. (to appear).

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