APPROXIMATION BY STEP FUNCTIONS

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1. Introduction. In a recent paper [5] I studied the Chebyshev approximation problem

\[ f \sim \sum_{i=1}^{p} x_i \chi_i, \]

i.e., the approximation of a given bounded real function \( f \) on a set \( A \) by linear combinations of given characteristic functions \( \chi_1, \ldots, \chi_p \) of subsets \( A_1, \ldots, A_p \) of \( A \) in the sense of minimizing the norm

\[ \| f - \sum x_i \chi_i \| = \sup \{ | f(a) - \sum x_i \chi_i(a) | : a \in A \} \]

by a proper choice of the \( x_i \). As the problem is one of linear programming namely to find \( x_i \) and \( s \) such that

\[ -s \leq f(a) - \sum x_i \chi_i(a) \leq s \quad \text{for all } a \in A \]

and such that \( s \) is minimal, several methods to get a solution are at hand. Here we are concerned with a method which is especially adapted to the problem and which in case of the “matrix problem”

\[ a_{ik} \sim x_i + y_k, \]

i.e., of approximating a given matrix \( (a_{ik}) \) by a matrix of the particular type \( (x_i + y_k) \), has proved to be very efficient. It is the “leveling process” [1; 2; 3] which roughly speaking for problem (1) consists in an alternatively repeated minimizing within the sets \( A_i \) (in problem (2) the rows and columns). In [4] I pointed out by an example that the effectiveness of the leveling process depends on the structure of the covering of \( A \) by the \( A_i \)'s and in [5] a decisive combinatorial property of the covering was introduced. The theorem which shows the bearing of this property on the approximation problem is here stated in the form of a necessary and sufficient condition. The examples given below disclose the surprising fact that approximation problems of the simple type as

\[ a_{ijk} \sim x_i + y_j + z_k, \]
\[ a_{ijk} \sim x_{ik} + y_{jk} + z_{ki}, \]
\[ a_{jk} \sim x_j + y_k + z_{j+k}, \]

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do not possess the property in question if the index sets are sufficiently large and therefore may be insensible to the leveling process.

2. For simplification we consider a finite set \( A \) and a covering \( \Gamma = \{ A_i : i \in I \} \) of \( A \) by a finite family of subsets \( A_i \) of \( A \), \( A = \bigcup_{i \in I} A_i \), \( I = \{ 1, \cdots, p \} \). With this covering is associated the family \( \Phi \) of all linear combinations

\[
\phi = \sum x_i \chi_i
\]

where \( \chi_i \) denotes the characteristic function of \( A_i \) and the \( x_i \)'s are real numbers. We consider the Chebyshev approximation of a given function \( f_0 \mid A \) by functions \( \phi \) of \( \Phi \). If we define the norm \( \| f \| = \max \{ |f(a)| : a \in A \} \) for \( f \mid A \) we have to find a \( \phi_0 \in \Phi \) such that

\[
\| f_0 - \phi_0 \| \leq \| f_0 - \phi \| \quad \text{for all } \phi \in \Phi.
\]

The leveling process presents itself if we reformulate the problem. Let us say that \( f \mid A \) and \( g \mid A \) are equivalent (with respect to the given covering \( \Gamma \)) if \( f - g \in \Phi \) then our problem is this: Given a function \( f_0 \); find an equivalent one, say \( f^* \), with least norm. For if \( f^* \) is equivalent to \( f_0 \) and of least norm we have \( f_0 = f^* + \phi_0 \) with \( \phi_0 \in \Phi \) and \( \phi_0 = f_0 - f^* \) is a solution of the approximation problem. So we have to work within the equivalence class of \( f_0 \) towards functions of smaller and smaller norms. The simplest way to produce a function equivalent to \( f \) is the transition \( f \rightarrow f + \gamma \chi_i \). If we take for \( \gamma \) the value

\[
\gamma_0 = -\frac{1}{2}(\max f \mid A_i + \min f \mid A_i),
\]

we have done the best for decreasing the norm. The transition

\[
f \rightarrow f^{(\gamma)} = f - \frac{1}{2}(\max f \mid A_i + \min f \mid A_i) \chi_i
\]

is called the leveling of \( f \) on \( A_i \). We evidently have \( \| f^{(\gamma)} \| \leq \| f \| \). So leveling is a step towards a solution and so it seems quite natural to apply iterations of the leveling on the different \( A_i \) alternatively and in some periodic fashion.

3. We start with \( f_0 \mid A \), define

\[
I f = ( \cdots ((f^{(1)})^{(2)}) \cdots )^{(p)}
\]

and

\[
f_n = L(f_{n-1}), \quad n = 1, 2, \cdots.
\]

Because of \( \| f_{n+1} \| \leq \| f_n \| \) we have the existence of \( \lim_n \| f_n \| = b \). The surprising fact is that \( b \) may be larger than

\[
\inf \{ \| f_0 - \phi \| : \phi \in \Phi \},
\]

the minimal approximation error, a defect which eventually cannot be
repaired by a rearrangement in the order of the different levelings. This is shown by the following example (there are simpler ones [4] but we use the one here for another reason):

Let \( A \) be the set \( \{1, 2, \ldots, 8\} \) and \( A_i \) be

\[
\begin{align*}
A_1 &= \{1, 2\}, & A_2 &= \{3, 4\}, & A_3 &= \{5, 6\}, & A_4 &= \{7, 8\}; \\
A_1' &= \{1, 3\}, & A_2' &= \{4, 5\}, & A_3' &= \{2, 7\}, & A_4' &= \{6, 8\}; \\
A_1'' &= \{3, 5, 7\}, & A_2'' &= \{1, 4, 8\}, & A_3'' &= \{2, 6\}.
\end{align*}
\]

Consider the function \( f_0 \) with the values \( f_0(1) = f_0(4) = f_0(6) = f_0(7) = 100 \) and the value \(-100\) on all other places. Then \( f_0 \) is already leveled on each \( A_i \). So leveling is ineffective. But we can get a function \( f \) equivalent to \( f_0 \) with smaller norm by adding to each place on the sets \( A_1', \ldots, A_4'' \) in the same order as listed above the values

\[
31, 25, 19, 9; \quad -24, -18, -10, 0; \quad 0, -8, -10,
\]

and for the resulting function \( f \) we find \( \|f\| = 99 \).

4. The inefficiency of the leveling process depends on the structure of the covering. Because the convergence of the sequence \( f_n \) of §3 is a highly intricate matter—but knowing that if there is convergence the limit function is leveled on all \( A_i \) and equivalent to the original function—we ask an intermediate question: Under what conditions on the covering are we allowed to conclude that a function \( g \), equivalent to \( f_0 \) and leveled on all \( A_i \), i.e., \( g^{(i)} = g \) for \( i \in I \), is of least norm? To give an answer to this question we define:

A function \( \sigma \mid A \) is said to be an AS-function ("function of alternating sign") with respect to the covering \( \Gamma = \{ A_i : i \in I \} \) if \( \sigma \) is not identically zero and

1. \( \sigma(a) \in \{-1, 0, 1\} \) for all \( a \in A \);
2. whenever \( \sigma \mid A_i \neq 0 \) there are at least two points \( x_1, x_2 \) on \( A_i \), with \( \sigma(x_1) = 1 \) and \( \sigma(x_2) = -1 \), \( i \in I \).

A covering \( \Gamma \) is called an L-covering (the L simply indicates the reference to the leveling process) if to each AS-function \( \sigma \mid A \) there is a function \( s \mid A \) not identically zero and satisfying

(L') sign \( s(x) \in \{0, \sigma(x)\} \) for all \( x \in A \);
(L'') \( \sum_{x \in A_i} s(x) = 0 \) for \( i \in I \).

With these definitions we can state the

**Theorem.** If \( \{ A_i : i \in I \} \) is an L-covering of \( A \) then each function \( g \) equivalent to \( f \) and leveled on each \( A_i \) yields in \( \phi = f - g \) a best Chebyshev approximation of \( f \) by linear combinations of the characteristic functions of the \( A_i \)'s. And conversely, if this is true for any \( f \) then \( \{ A_i : i \in I \} \) is an L-covering.
Proof.

1. Let us assume that there is a function $g$ leveled on all $A_i$ and satisfying $\|g\| > \|f\|$ for some function $f$ equivalent to $g$. We are going to show that $\Gamma = \{A_i : i \in I\}$ is no $L$-covering. Let $\|g\| = a$, then $a > 0$ and

$$\sigma(x) = \begin{cases} +1 & \text{if } g(x) = a, \\ -1 & \text{if } g(x) = -a, \\ 0 & \text{elsewhere} \end{cases}$$

defines an $A^S$-function. With $f = g + \sum y_i \chi_i$ we get the inequalities

$$\sum y_i \chi_i(x) < 0 \text{ for } \sigma(x) = 1, \quad \sum y_i \chi_i(x) > 0 \text{ for } \sigma(x) = -1.$$ 

Now assume that $\Gamma$ is an $L$-covering. Then there is a function $s|A$ not identically zero and satisfying $(L')$ and $(L'')$. This gives

$$\sum_{x \in A} \left( \sum_{i \in I} y_i \chi_i(x) \right) s(x) < 0.$$ 

The left side may be written

$$\sum_i y_i \sum_{x \in A} \chi_i(x) s(x) = \sum_i y_i \sum_{x \in A_i} s(x) = 0.$$ 

This is a contradiction and the sufficiency of the condition is proved.

2. Now let us assume that for any $g$ leveled on all $A_i$ equivalent to $f$ we have $\|g\| \leq \|f\|$. Then for any $A$ function $\sigma|A$ and any numbers $y_i$ we have

$$\|\sigma + \sum y_i \chi_i\| \geq 1.$$ 

Define $\sigma_i(x) = \sigma(x) \chi_i(x)$ and $A' = \{x : x \in A \text{ and } \sigma(x) \neq 0\}$. Then

$$(*) \text{ not all the numbers } \sum y_i \sigma_i(x), x \in A', \text{ are of the same sign.}$$

For if for instance all these numbers would be $< 0$ then with some $\rho > 0$ we could replace the $y_i$ by $y_i' = \rho y_i$ and arrive at $\|\sigma + \sum y_i' \chi_i\| < 1$ in contradiction to (6). But (*) is a well-known [6] sufficient condition that the system

$$\sum_{x \in A} \sigma_i(x) S(x) = 0, \quad S \geq 0$$

allows a solution $S$ not identically zero. With $s(x) = \sigma(x) S(x)$ we see that $(L')$, $(L'')$ can be satisfied with $s \neq 0$.

5. Examples of $L$-coverings.

Proposition 1. For every $n \times m$-matrix the system of rows and columns is an $L$-covering.
Proof. Every AS-function \( \sigma \) on the matrix array contains an irreducible AS-function \( \sigma' \) which on a row or column where it is not zero yields exactly one +1 and one -1. Evidently \( \sigma' \) is a function \( s \) fitting to \( \sigma \) in the sense of the theorem above.

Proposition 2. If \( \{ A_i : i \in I \} \) is an L-covering of \( A \) and \( B \subset A \) then the "trace covering" on \( B \), \( \{ B \cap A_i : i \in A \text{ and } B \cap A_i \neq \emptyset \} \) is an L-covering of \( B \).

Proof. Let \( \sigma \) be an AS-function with respect to the trace covering. We extend it by defining \( \sigma| (A - B) = 0 \) and get an AS-function with respect to \( \{ A_i : i \in I \} \). By our theorem we have a function \( s|A \) fitting to \( \sigma|A \). Evidently \( s|B \) fits to \( \sigma|B \) with respect to the trace covering.

6. As a matter of fact there are many simple coverings which are no L-coverings.

The coverings belonging to the approximation problems (3), (4) and (5) are no L-coverings if the index sets are large enough.

Proof.

Concerning (3). The covering of the example in §3 can be considered as a trace covering on a cubic \( 4 \times 4 \times 3 \)-matrix covered by its 2-dimensional layers. So by Proposition 2 a cubic matrix with at least four 2-dimensional layers in each direction is no L-covering.\(^1\)

Concerning (4). Consider the following AS-function \( \sigma \) on a cubic \( 5 \times 5 \times 6 \)-matrix

| \( \begin{array}{cccc}
1+ & 4+ & 1- & 4- \\
2- & 5- & 3+ & 2+ \\
\hline
6- & 2+ & 3- & 5+ \\
2+ & 5+ & 6+ & 2- \\
\hline
3- & 4+ & 3+ & 6- \\
6+ & 3- & 5- & 4+ \\
\hline
1+ & 3- & 1- \\
3- & 2+ & 2+ \\
\hline
1- & 4- & 1- & 5+ \\
3+ & 5+ & 5+ & 4- \\
\end{array} \) |

\(^1\) This fact disproves a hypothesis of M. Golomb [3, p. 324, (10.53)].
where the figure indicates the height of the layer and the sign behind it the sign of $\sigma$. On all other places $\sigma$ has the value 0. It is easy to check that any function $s$ satisfying $(L')$ and $(L'')$ with respect to $\sigma$ is identically zero. So the covering of problem (4) is no $L$-covering.

Concerning (5). We use the preceding example. We project its cubic matrix array into a $(j, k)$-plane in such a way that the rods of the matrix are projected into the lines $j = \text{const.}, k = \text{const.},$ and $j + k = \text{const.}$ and that no two rods have colinear images. So we see that the covering of the problem (4) is a trace covering of problem (5). Proposition 2 again proves that the covering of (5) is no $L$-covering as soon as there are sufficiently many layers in each family.

References