OPEN 3-MANIFOLDS WHICH ARE SIMPLY CONNECTED AT INFINITY

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A triangulated open manifold $M$ will be called 1-connected at infinity if each compact subset $C$ of $M$ is contained in a compact polyhedron $P$ in $M$ such that $M - P$ is connected and simply connected. Stallings has shown that, if $M$ is a contractible open combinatorial manifold which is 1-connected at infinity and is of dimension $n \geq 5$, then $M$ is piecewise-linearly homeomorphic to Euclidean $n$-space $E^n$ [5].

**Theorem 1.** Let $M$ be a contractible open 3-manifold, each of whose compact subsets can be imbedded in $E^3$. If $M$ is 1-connected at infinity, then $M$ is homeomorphic to $E^3$.

Notice that, in order to prove the 3-dimensional Poincaré conjecture, it would suffice to prove Theorem 1 without the hypothesis that each compact subset of $M$ can be imbedded in $E^3$. For, if $M$ is a simply connected closed 3-manifold and $p$ is a point of $M$, then $M - p$ is a contractible open 3-manifold which is clearly 1-connected at infinity. Conversely, if the 3-dimensional Poincaré conjecture were known, then the hypothesis that each compact subset of $M$ can be imbedded in $E^3$ would be unnecessary.

All spaces and mappings in this paper are considered in the polyhedral or piecewise-linear sense, unless otherwise stated. As usual, by an open $n$-manifold is meant a noncompact connected space triangulated by a countable simplicial complex without boundary, such that the link of each vertex is piecewise-linearly homeomorphic to the usual $(n-1)$-sphere.

**Proof of Theorem 1.** Let $X$ be an arbitrary compact subset of $M$, and let $Y$ be a connected compact subset of $M$ which contains $X$. Using the fact that $M$ is 1-connected at infinity, choose a compact polyhedron $P$ such that $Y \subset P \subset M$ with $M - P$ connected and simply connected. If $N$ is a regular neighborhood of $P$ which contains $P$ in its interior, then $M - N$ is also connected and simply connected, and the component $N_0$ of $N$ which contains $Y$ is a (connected) compact orientable 3-manifold with boundary [6]. Since $M$ is contractible, and since the boundary $S_0$ of $N_0$ separates $M$ into exactly two com-

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ponents $\text{Int } N_0$ and $M - N_0$, $S_0$ is connected by Alexander duality.

Since $M - N$ is simply connected, the 1-dimensional Betti number of $N$ is zero (also by Alexander duality). But the 1-dimensional Betti number of a bounded orientable 3-manifold is at least as large as the sum of the genera of its boundary surfaces \cite[p. 223]{4}. It follows that $S_0$ is a 2-sphere. The assumption that each compact subset of $E^3$ can be imbedded in $E^3$ now implies that $N_0$ is a 3-cell which contains $X$ in its interior.

Since each compact subset of $M$ lies interior to a 3-cell in $M$ it follows easily that $M$ is the union of a sequence \{$C_n$\}_n of 3-cells, with $C_n \subseteq \text{Int } C_{n+1}$, $n = 1, 2, \cdots$. A theorem of Brown now applies to show that $M$ is homeomorphic to $E^3$ \cite{1}.

Theorem 1 will next be generalized by suppressing the restriction that $M$ be contractible and relaxing the restriction that it be 1-connected at infinity. Let an open manifold $U$ be called simply connected at infinity if each compact subset $B$ of $U$ is contained in a compact polyhedron $Q$ in $U$ such that each component of $U - Q$ is simply connected. By a punctured cube will be meant a space obtained from a 3-sphere by deleting the interiors of a finite (positive) number of mutually disjoint polyhedral 3-cells.

**Lemma 1.** Let $U$ be an open 3-manifold which is simply connected at infinity, and such that each compact subset of $U$ can be imbedded in $E^3$. Then each compact subset of $U$ lies interior to a punctured cube in $U$.

**Proof.** Let $A$ be an arbitrary compact subset of $U$, and let $B$ be a connected compact subset of $U$ containing $A$. Since $U$ is simply connected at infinity, there is a compact polyhedron $Q$ in $U$ containing $B$, such that each component of $U - Q$ is simply connected. If $N$ is a regular neighborhood of $Q$ containing $Q$ in its interior, then each component of $U - N$ is also simply connected, and the component $N_0$ of $N$ which contains $B$ is a (connected) compact orientable 3-manifold with boundary \cite{6}.

It will be shown that the fact that each component of $U - N$ is simply connected implies that $N_0$ lies in a punctured cube in $U$. The proof of this is by induction on the sum $g$ of the genera of the boundary surfaces of $N_0$. If $g = 0$, then the hypothesis that each compact subset of $U$ can be imbedded in $E^3$ implies that $N_0$ itself is a punctured cube. Now assume that the conclusion follows if $g < k$, where $k \geq 1$, and let the sum of the genera of the boundary surfaces of $N_0$ be $k$.

If $S$ is a closed orientable 2-manifold of positive genus on the boundary of $N_0$, let $J$ be a simple closed curve encircling one of the handles of $S$. Then the Dehn lemma \cite{2} gives a 2-cell $D$ with $\text{Bd } D = J$.
and \( \text{Int } D \subset U - N \), since each component of \( U - \text{Int } N \) is simply connected.

It is first shown that each component of \( V = (U - N) - D \) is simply connected. Let \( K \) be a simple closed curve in \( V \) and let \( f \) be a piecewise-linear map of a 2-cell \( E \) into \( U - N \) such that \( f|\text{Bd } E \) is a homeomorphism of \( \text{Bd } E \) onto \( K \) and such that \( f \) is in general position with respect to \( D \), in the sense that each component of \( f^{-1}(D) \) is a simple closed curve. Let \( K^1 \) be an “inner” one of these simple closed curves, bounding the subdisk \( E^1 \) of \( E \). Then \( K^1 \) can be eliminated by first redefining \( f \) on \( E^1 \) and then deforming the new image of \( E^1 \) slightly away from \( D \). After a finite number of steps of this kind, it is seen that \( K \) can be shrunk to a point in \( V \).

Now thicken the 2-cell \( D \) to form a 3-cell \( C \) such that \( S \cap \text{Bd } C \) is an annular ring \( R \) with \( C - R \subset U - N \). Then each component of \( U - (N \cup C) \) is simply connected, and the sum of the genera of the boundary surfaces of \( N \cup C \) is \( k - 1 \). By induction \( U \) therefore contains a punctured cube containing \( N \cup C \) and hence containing \( A \) in its interior.

The following elementary lemma is easily proved.

**Lemma 2.** Let \( A \) and \( B \) be punctured cubes with \( A \subset \text{Int } B \) and let \( C \) and \( D \) be 3-cells with \( C \subset \text{Int } D \). Suppose that \( S \) and \( T \) are components of \( \text{Bd } A \) and \( \text{Bd } B \), respectively, such that \( S \) separates \( \text{Int } A \) and \( T \) in \( B \). If \( f \) is a piecewise-linear homeomorphism of \( A \) into \( C \) such that \( f(S) = \text{Bd } C \), then there is a piecewise-linear homeomorphism \( g \) of \( B \) into \( D \) such that \( g(T) = \text{Bd } D \) and \( g/A = f \).

**Theorem 2.** Let the open 3-manifold \( U \) be the union of a sequence \( \{A_i\}_i \) of punctured cubes, with \( A_i \subset \text{Int } A_{i+1}, \ i = 1, 2, \ldots \). Then there is a totally disconnected subset \( Y \) of \( E^3 \) such that \( U \) and \( E^3 - Y \) are homeomorphic.

**Proof.** The collection \( \{A_i\}_i \) is first subjected to a sequence of alterations as follows. In the first step, a new punctured cube \( A_1^1 \) interior to \( A_2 \) is obtained from \( A_1 \) by adding to \( A_1 \) each component of \( A_2 - A_1 \) which contains no component of \( \text{Bd } A_2 \) (the closure of each such component of \( A_2 - A_1 \) is a 3-cell). Each component of \( \text{Bd } A_1^1 \) will then separate \( \text{Int } A_1^1 \) and some component of \( \text{Bd } A_2 \) in \( A_2 \).

In the second step, the punctured cube \( A_2^2 \) interior to \( A_3 \) is obtained from \( A_2 \) by adding to \( A_2 \) each component of \( A_3 - A_2 \) which contains no component of \( \text{Bd } A_3 \), and then \( A_2^2 \) is obtained from \( A_1^1 \) by adding to \( A_1^1 \) each component of \( A_2^2 - A_1^1 \) which contains no component of \( \text{Bd } A_2^2 \). Now each component of \( \text{Bd } A_2^2 \) separates \( \text{Int } A_1^1 \) from some...
component of $\text{Bd } A_3^2$ in $A_3^2$, and each component of $\text{Bd } A_n^2$ separates $\text{Int } A_n^2$ and $\text{some component of } \text{Bd } A_n$ in $A_n$.

Suppose that the punctured cubes $A_i^{i-1}, A_2^{i-1}, \ldots, A_{n-1}^{i-1}$ are the result of the first $i-1$ steps in this process. In the $i$th step the punctured cube $A_i^i$ is obtained from $A_i$ by adding to $A_i$ each component of $A_{i+1}^i - A_i$ which contains no component of $\text{Bd } A_{i+1}$; then $A_{i-1}^i$ is obtained from $A_{i-1}^{i-1}$ by adding to $A_{i-1}^{i-1}$ each component of $A_i^i - A_{i-1}^i$ which contains no component of $\text{Bd } A_i^i$, and so on; finally the punctured cube $A_1^1$ is obtained from $A_1$ by adding to $A_1$ each component of $A_2^1 - A_1$ which contains no component of $\text{Bd } A_2^1$. Now the punctured cubes $A_i^1, \ldots, A_1^1$ satisfy the condition that each component of $\text{Bd } A_i, j<i$, separates $\text{Int } A_j^i$ and some component of $\text{Bd } A_{i+1}$ in $A_j$. This process is continued by induction. Notice that $A_m^n = A_n^n$ if $m$ and $n$ are sufficiently large, $i=1, 2, \ldots$. Consequently the result of this sequence of alterations is a new sequence $\{B_i\}_{i=1}^\infty$ of punctured cubes such that (1) $U = \bigcup_{i=1}^\infty B_i$, (2) $B_i \subset \text{Int } B_{i+1}$ for each $i$, and (3) each component of $\text{Bd } B_i$ separates $\text{Int } B_i$ and some component of $\text{Bd } B_{i+1}$.

Now let $E^3$ be expressed as the union of a sequence $\{C_i\}_{i=1}^\infty$ of polyhedral 3-cells such that $C_i \subset \text{Int } C_{i+1}$, $i=1, 2, \ldots$. Let $S_i$ be any component of $\text{Bd } B_i$ and, $S_{i-1}$ having been defined as a component of $\text{Bd } B_{i-1}$, let $S_i$ be a component of $\text{Bd } B_i$ such that $S_{i-1}$ separates $\text{Int } B_{i-1}$ and $S_i$ in $B_i$.

Then use Lemma 2 to define by induction a sequence $\{g_i\}_{i=1}^\infty$ of maps such that, for each $i$, (1) $g_i$ is a piecewise-linear homeomorphism of $B_i$ into $C_i$, (2) $g_i(S_i) = \text{Bd } C_i$, and (3) $g_i|_{B_{i-1}} = g_{i-1}$. Finally define a homeomorphism $f$ of $U$ into $E^3$ by setting $f(x) = g_i(x)$ if $x \in B_i$.

If $F_i$ is the closure of $C_i - g_i(B_i)$, then clearly $F_i$ is the union of a finite number of mutually disjoint 3-cells. If $X = \bigcap_{i=1}^\infty F_i$, then $f(U) = E^3 - X$. Now let $G$ be the decomposition space obtained from $E^3$ by shrinking each component of $X$ to a point, and let $h$ be the natural map of $E^3$ onto $G$, which is a homeomorphism on $E^3 - X$. Then $Y = h(X)$ is a totally disconnected subset of $G$, and it follows from [3] that $G$ is homeomorphic to $E^3$. But $hf$ is a homeomorphism of $U$ onto $G - Y$, so the proof of Theorem 2 is complete.

**Theorem 3.** Let $U$ be an open 3-manifold, each of whose compact subsets can be imbedded in $E^3$. If $U$ is simply connected at infinity, then there is a totally disconnected subset $Y$ of $E^3$ such that $U$ and $E^3 - Y$ are homeomorphic.
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Proof. Lemma 1 implies that U can be expressed as the union of an increasing sequence of punctured cubes, as in the hypotheses of Theorem 2.

References


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