STELLAR NEIGHBORHOODS IN POLYHEDRAL
MANIFOLDS

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Introduction. It is the purpose of this paper to give some weaker
analogues for polyhedral manifolds of well-known properties of com-
binatorial manifolds. This is now possible primarily because of the
recent work of Mazur [3; 4] and M. Brown [1]. A crucial issue in this
area is the extent to which these analogues may be improved for
arbitrary triangulated manifolds.

We shall prove a theorem which will be applied later on, after
making some preliminary definitions. The join of two spaces X and
Y is represented by X o Y. A map f of X o Y — Y on itself is called
ray preserving if for each ray p o q — q, p in X, q in Y, f(p o q - q)
\subset p o q — q. A subset K of a cone X o p is called starlike if p \in K
and each segment p o x, x \in X, meets K in a connected set; p is the
center of K. Consider E^n to be the open cone over S^{n-1} from the origin
p, in the usual way. We coordinatize E^n by (x, t), x \in S^n, t a real
number with 0 \leq t < \infty; (S^{n-1}, 0) = p. Let D^n be the unit n-ball
p o (S^{n-1}, 1).

Theorem 1. Let K be a compact starlike set with center p, lying in
the interior of D^n. There is a ray preserving map f of E^n on itself such
that f(K) = p, f| E^n - K is one-to-one and f| E^n - D^n is the identity.

Proof. Let \rho be the usual euclidean metric for E^n. Choose 0 < \epsilon < 1
so that K \subseteq p o (S^{n-1}, \epsilon) = D_\epsilon. First we define f on D_\epsilon. If z = (x, t) \in D_\epsilon,
f(z) = (x, \rho(z, K)). It is clear that f is a ray preserving map and mono-
tone nonincreasing with respect to the coordinate t, so f(D_\epsilon) \subseteq D_\epsilon;
furthermore f(K) = p and f| D_\epsilon - K is one-to-one. To see the last
property suppose (x, t_1) and (x, t_2) are points of D_\epsilon - K with t_1 < t_2.
Let y \in K so that \rho((x, t_2), y) = \rho((x, t_2), K). Since y o p \subseteq K and \rho
is the ordinary straight line distance, obviously \rho((x, t_1), y o p)
< \rho((x, t_2), y) and accordingly f(x, t_1) \neq f(x, t_2).

We next extend f to the annulus D^n - D_\epsilon, so that f is fixed on Bd D^n.
If z = (1 - \tau) \cdot (x, \epsilon) + \tau \cdot (x, 1), 0 \leq \tau \leq 1 and x \in S^n, let f(z) = (1 - \tau)
\cdot f(x, \epsilon) + \tau \cdot (x, 1). Finally let f be fixed outside of D^n. Now f has the

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desired properties and is also monotone nonincreasing in the $t$-coordinate. Clearly $f$ is realizable as the end of a ray preserving pseudo-isotopy starting at the identity, which is fixed outside of $D^n$ and is monotone nonincreasing in $t$.

**Corollary.** $f$ is a homeomorphism of $E^n - K$ onto $E^n - p$.

**Theorem 2.** Suppose $X$ is a compact metric finite dimensional space and $K$ is a compact starlike set in $X \circ p - X$. There is a ray preserving map $f$ of $X \circ p$ on itself so that $f(K) = p$, $f$ is a homeomorphism on $X \circ p - K$ and $f$ is fixed on a neighborhood of $X$.

**Proof.** Suppose $X$ is imbedded in $(S^{n-1}, 1) \subseteq E^n$. This is extended in the obvious way to an imbedding of $X \circ p$ in $D^n$. $K$ is now starlike in Int $D^n$ so we apply Theorem 1 here. Since the map we then get is ray preserving and maps $D^n$ on itself, clearly it maps $X \circ p$ on itself. The restriction of this map to $X \circ p$ satisfies the necessary conditions except possibly the last, however a slight alteration can be made in the proof of Theorem 1 so that the map constructed therein is also fixed on some neighborhood of Bd $D^n$.

1. A theorem due to Mazur. In [3] Mazur indicated that Theorem 4 of this section would follow from the form of the Generalized Schoenflies Theorem later proved by M. Brown [1] and Morse [6]; he also gave a proof of Theorem 5 in [4]. In [7] we indicated a simpler proof of the latter, using Theorem 4. For completeness we include our own proofs.

**Theorem 3 (Moise-Alexander).** Let $M$ be a compact Hausdorff space which is the union of two open sets, each of which is homeomorphic to $E^n$. Then $M$ is homeomorphic to $S^n$.

This was conjectured by Alexander for $n = 3$ and proved by Moise in that case [5]. Many people, including the author, have independently noticed that the proof for general $n$ follows from [1] in a manner quite similar to Moise’s proof.

**Theorem 4 (Mazur).** If $Y \circ p$ is locally $n$-euclidean at $p$, then the suspension $S(Y)$ is homeomorphic to $S^n$.

**Proof.** Consider $S(Y)$ as $Y \times [-1, 1]$ with $Y_{-1}$ and $Y_1$ identified as distinct points $p'$ and $p$, respectively. Suppose $U$ is a euclidean neighborhood of $p$ in $S(Y)$ so that $\overline{U}$ is compact. For some $t_0$, $-1 < t_0 < 1$, $U(Y_{t_0}) \subseteq [-1, 1] \subseteq U$. $Y_{t_0}$ is closed in $S(Y)$, so $Y$, hence

*A more general theorem can be proved but this is sufficient for our applications.*
S(Y), is compact. Let \( h \) be a homeomorphism of \( S(Y) \) on itself, fixed on \( Y_0, \) which takes \( U(Y; t_0 \leq t) \) onto \( U(Y; t \leq t_0). \) \( U \) and \( h(U) \) are two euclidean neighborhoods which cover \( S(Y). \)

**Corollary.** \( Y \circ p - Y \) is homeomorphic to \( E^n. \)

**Proof.** There is a simple homeomorphism of \( Y \circ p - Y \) onto \( U(Y; 0 < t) \) in \( S(Y) \) and a homeomorphism of the second set onto \( S(Y) - p'. \)

**Theorem 5 (Mazur).** Let \( v \) be a vertex of a triangulated \( n \)-manifold \( M. \) The open star of \( v \) in \( M \) is homeomorphic to \( E^n. \)

**Proof.** Let \( B \) be the link of \( v \) in \( M. \) Since \( B \circ v - B \) is a neighborhood of \( v \) in \( M, \) by Theorem 4 \( S(B) \approx S^n. \) (We use the notation \( X \approx Y \) if \( X \) and \( Y \) are homeomorphic spaces.) Therefore by the corollary, \( B \circ v - B \approx E^n. \)

2. **Some analogues to combinatorial theorems.** M. Brown has called a subset of an \( n \)-manifold **cellular** if it is the intersection of a sequence \( (C_n) \) of topological \( n \)-cells where \( C_{n+1} \subseteq \text{Int} \ C_n. \) Let \( Q \) be a subpolyhedron of a polyhedron \( P. \) The **stellar neighborhood of \( Q \) in \( P \) is the union of all open simplexes of \( P \) which have vertices lying in \( Q. \) Star \( Q \) is the union of all closed simplexes of \( P \) which meet \( Q. \) \( Q \) is said to be **full** in \( P \) if it contains each simplex of \( P, \) all of whose vertices lie in \( Q. \)

Henceforth all simplexes will be considered closed. If \( \sigma \) is a simplex of \( P, \) by \( \sigma^* \) we mean the union of all simplexes of \( P \) which contain \( \sigma, \) or equivalently \( \sigma^* = \sigma \circ \text{Lk} \sigma \) — where \( \text{Lk} \sigma \) is the link of \( \sigma \) in \( P; \sigma^* \) is called the **cluster** of \( \sigma. \) \( \hat{\sigma} \) is the union of all proper faces of \( \sigma; \sigma^b = \sigma - \hat{\sigma}. \) Finally \( \beta(\sigma) \) will be the barycenter of \( \sigma. \)

**Theorem 6.** Let \( K \) be a full finite subpolyhedron of a polyhedral \( n \)-manifold \( M. \) Then \( K \) is cellular in \( M \) if and only if its stellar neighborhood is homeomorphic to \( E^n. \)

**Proof.** (a) Assume \( K \) is cellular. Let \( V \) be the stellar neighborhood of \( K, \) \( B = \text{Bd} V \) and \( N = \text{Star} K = \overline{V}. \) Clearly \( B \neq \emptyset. \) Let \( X \) be the set of all midpoints of straight segments joining points of \( B \) to points of \( K. \) We shall show that the decomposition space of \( V \) with \( K \) identified to a point is homeomorphic to the open cone over \( X. \) For since \( K \) is full, each simplex of \( N \) can be represented as a join \( \sigma \circ \tau, \sigma \subseteq K, \tau \subseteq B. \) When \( K \) is identified to a point, \( \sigma \circ \tau - \sigma \) becomes simply the open cone over the set of all midpoints of segments from \( \tau \) to \( \sigma; \) the vertex of the cone is the image of \( K \) in the decomposition space. Since
such a representation holds for each maximal simplex in \( N \), the assertion is easily established.

Now let \( C \) be a topological \( n \)-cell lying in \( V \) with \( K \subset \text{Int} \ C \). By Theorem 1 of [1] there is a map \( f \) of \( V \) on itself fixed outside of \( C \) such that \( f(K) \) is a point and \( f|_{V-K} \) is one-to-one. Hence the existence of the map \( f \) shows that \( V \) is homeomorphic to the open cone over \( X \), with vertex \( f(K) \). By Theorem 4 and its corollary, \( V \approx E^n \).

(b) Now suppose \( V \approx E^n \). Since \( K \) is full we can conclude from the argument in part (a) that \( V \) can be represented as an open mapping cylinder over \( K \) (from the space \( X \)) using the segments from \( B \) to \( K \) minus their endpoints in \( B \). Since \( K \) is compact there is a topological \( n \)-cell \( C_1 \) in \( V \) so that \( K \subset \text{Int} \ C_1 \). Now let each segment be linearly parametrized from 0 to 1 with 0 the parameter of the endpoint in \( K \). There is an \( \epsilon_1, 0 < \epsilon_1 < 1/2 \), so that on each segment the subsegment from 0 to \( \epsilon_1 \) lies in \( \text{Int} \ C_1 \). Hence let \( h_1 \) be a homeomorphism of \( V \) into \( \text{Int} \ C_1 \), fixed on \( K \), which maps each ray from 0 to 1 linearly onto its subray from 0 to \( \epsilon_1 \). Thus there is a topological \( n \)-cell \( C_1 \) in \( \text{Int} \ C_1 \) with \( K \subset \text{Int} \ C_2 \). The process may be continued by induction to define a sequence \( (C_n) \) of \( n \)-cells having the properties that \( \bigcap_n C_n = K \) and \( C_{n+1} \subset \text{Int} \ C_n \).

**Corollary.** Let \( K \) be a finite subpolyhedron of a polyhedral \( n \)-manifold \( M \). Then \( K \) is cellular if and only if its first barycentric stellar neighborhood in \( M \) is homeomorphic to \( E^n \).

**Proof.** The first barycentric subdivision of \( K \) is full in the corresponding subdivision of \( M \) by Lemma 9.4, p. 71 of [2].

It may be noted that Theorem 6 generalizes Theorem 5 since vertices are clearly cellular.

**Theorem 7.** Let \( \sigma \) be a simplex in a polyhedral \( n \)-manifold \( M \). Then \( \text{Int} \ \sigma^* \) is homeomorphic to \( E^n \).

**Proof (Added in proof).** Adapting the notation of Alexander in [8] (our manifolds may be noncompact) let \( M = \sigma \circ \text{Lk} \ \sigma + R \) and consider the simple subdivision \( M \to \beta(\sigma) \circ (\sigma \circ \text{Lk} \ \sigma) + R \). By Theorem 5, \( E^n \) is homeomorphic to \( \text{Int}(\beta(\sigma) \circ (\sigma \circ \text{Lk} \ \sigma)) = \text{Int} \ \sigma^* \). This method also tells us that, by Theorem 4, \( S(\sigma \circ \text{Lk} \ \sigma) \approx S^n \). Hence we have the

**Corollary (Added in proof).** If \( \sigma \) is a \( k \)-dimensional simplex in a triangulated \( n \)-manifold then \( S^k \circ \text{Lk} \ \sigma \approx S^n \).

**Proof (Added in proof).** For \( S(\sigma \circ \text{Lk} \ \sigma) = S(\bar{\sigma}) \circ \text{Lk} \ \sigma \).
Theorem 8. Let $M$ be an $n$-manifold with triangulation $T$. Let $\sigma$ be a simplex of the first barycentric subdivision $T'$ of $T$. Then $\sigma$ is cellular in $M$.

Proof. Let $\tau$ be the lowest dimensional simplex of $T$ for which $\beta(\tau)$ is a vertex of $\sigma$. Obviously $\tau \subseteq \text{Int} \sigma^*$. We shall show that Star $\sigma \subseteq \tau^*$, Star $\sigma$ formed with respect to $T'$ and $\tau^*$, with respect to $T$. For any other vertex $\beta(\tau')$ of $\sigma$, $\tau' \supseteq \tau$; so if $\tau^n$ is an $n$-dimensional simplex containing $\beta(\tau')$, we have $\tau^n \supseteq \tau$ and hence $\tau^n \subseteq \tau^*$. It now may easily be seen that $\sigma \subseteq \text{Int} \tau^*$.

Again consider the simple subdivision $M = \tau \circ \text{Lk} \tau + R \rightarrow \beta(\tau) \circ (\hat{\tau} \circ \text{Lk} \tau) + R$. We can see that $\sigma$ is even a subcone, thus a compact starlike set in $\text{Int} (\beta(\tau) \circ (\hat{\tau} \circ \text{Lk} \tau)) = \text{Int} \tau^*$. Since $\text{Int} \tau^* \approx E^n$ and by Theorem 2 $\sigma$ is pointlike in $\text{Int} \tau^*$, it follows by Theorem 3 of [1] that $\sigma$ is cellular.

Corollary. The stellar neighborhood of $\sigma$ with respect to $T'$ is homeomorphic to $E^n$.

Proof. $\sigma$ is full in $T'$.

Theorem 9 (Added in proof). Let $M$ be an $n$-manifold with triangulation $T$. If $\sigma$ is a simplex of the second barycentric subdivision $T''$ of $T$ then $\sigma^*$ is cellular in $M$.

Proof (Added in proof). $T'$ will denote the first barycentric subdivision of $T$. Suppose $\sigma$ is a simplex of $T''$. Let $\tau$ be the lowest dimensional simplex of $T'$ whose barycenter $\beta(\tau)$ is a vertex of $\sigma$. As before, it follows that $\sigma^* \subseteq \tau^*$. It is also fairly easy to see that each vertex of $\tau$ lies in $\sigma^*$.

Now let $v$ be the minimal simplex of $T$ whose barycenter $\beta(v)$ is a vertex of $\tau$. We may conclude that $v$ is the minimal simplex of $T$ whose barycenter lies in $\sigma^*$. For if $v_1 \in T$ and $\beta(v_1) \in \sigma^*$, denote by $\sigma_1$ the simplex of $T''$ which contains $\sigma$ and $\beta(v_1)$. Since $\sigma_1$ must be $n$-dimensional let $\tau_1$ be the $n$-simplex of $T'$ containing $\sigma_1$. Inasmuch as $\sigma_1 \supseteq \sigma$, $\tau_1 \supseteq \tau$ and $\beta(\tau) \subseteq \sigma$, it may be seen that $\beta(v_1) \in \tau$. This proves that $v \subseteq v_1$.

Consider the simple subdivision $M = v \circ \text{Lk} v + R \rightarrow \beta(v) \circ P + R$, where $P = \hat{v} \circ \text{Lk} v$. Evidently $\sigma^* \subseteq \beta(v) \circ P$; we shall prove that $\sigma^* \subseteq \text{Int} \beta(v) \circ P$, the latter, of course, being a homeomorph of $E^n$.

Firstly, it may be established without too much difficulty, by a straightforward induction on $n$, that if $\sigma^* \subseteq \beta(v)^* \circ P$ with respect to $T'$ then $\sigma^n \cap P \subseteq \dot{v}$. Now suppose $\sigma^n \subset T''$ so that $\sigma^n \supseteq \sigma_1$ and $\sigma^n \cap \dot{v} \neq \emptyset$. Let $\tau^n \in T'$ with $\sigma^n \subseteq \tau^n$ and $\tau_1 = \tau^n \cap \dot{v}$. Then $\sigma^n$ contains $\beta(\tau_1)$ for
some face $\tau_0$ of $\tau_1$; this in turn necessitates that $\sigma^n$ contains a vertex $v$ of $\tau_0$. The vertex $v$ must be the barycenter of a face $v_0$ of $\tau$. Since $v \in \sigma^*$, this contradicts the minimality condition on $v$ proved above.

Finally we show that $\sigma^*$ is starlike in $\beta(v) \circ P$. It will then be obvious from our previous arguments that $\sigma^*$ is cellular in $M$. Let $\tau^n$ be any $n$-simplex of $T'$ which contains $\sigma$. By elementary analytic geometry it may be verified that $\sigma^* \cap \tau^n$ is convex in $\tau^n$. (See for example formula (2) on p. 62 of [2].) This completes the proof of the theorem.

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