MINIMAL REGULAR SPACES
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1. Introduction. If $\mathcal{P}$ is a property of topologies, a space $(X, 3)$ is minimal $\mathcal{P}$ if $3$ has property $\mathcal{P}$, but no topology on $X$ which is strictly weaker (= smaller) than $3$ has $\mathcal{P}$. Such spaces have been investigated for the case $\mathcal{P}=$ Hausdorff $[2; 5]$, a well-known result being that while every compact space is minimal Hausdorff, the converse is not true. We consider here the case $\mathcal{P}=$ regular; other properties are discussed by one of the authors in a paper to appear.

Filter-bases on spaces will be used extensively (for definitions not given here, see [1]). A filter-base is open (closed) if its elements are open (closed) sets. A filter-base will be called regular if it is open and is equivalent to a closed filter-base. The name is suggested by the fact that the filter-base of open neighborhoods of a point of a regular space is regular since it is equivalent to the filter-base of closed neighborhoods of that point.

2. Characterizations of minimal regular spaces. We will be concerned with spaces satisfying one or both of the following conditions:

(a) Every regular filter-base which has a unique adherent point is convergent.

(b) Every regular filter-base has an adherent point.

**Theorem 1.** A regular space which satisfies (a) also satisfies (b).

**Proof.** Suppose $\mathcal{B}$ is a regular filter-base on the regular space $(X, 3)$ and that $\mathcal{B}$ has no adherent point. Let $\mathcal{C}$ be a closed filter-base equivalent to $\mathcal{B}$. Fix $p \in X$ and let $\mathcal{U}$ and $\mathcal{V}$ be the filter-bases of open and closed neighborhoods of $p$, respectively. Since $3$ is regular, $\mathcal{U}$ and $\mathcal{V}$ are equivalent. Then $\mathcal{R} = \{ B \cup U : B \in \mathcal{B}, U \in \mathcal{U} \}$ is an open filter-base equivalent to the closed filter-base $\{ C \cup V : C \in \mathcal{C}, V \in \mathcal{V} \}$ and is therefore regular. It is clear that $p$ is the unique adherent point of $\mathcal{R}$ and that $\mathcal{R}$ does not converge to $p$. This denial of the hypothesis establishes the theorem.

**Theorem 2.** In order that a regular space be minimal regular, it is necessary and sufficient that it satisfy (a).

**Proof.** Suppose $(X, 3)$ is regular and that $\mathcal{B}$ is a regular filter-base having the unique adherent point $p$ to which it does not converge.

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1 As used in this paper, the condition of regularity includes $T_1$ separation, i.e., singletons are closed.

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For each $x \in X$, let $\mathcal{U}(x)$ be the filter-base of $\mathcal{O}$-open neighborhoods of $x$ and define $\mathcal{U}'(x) = \mathcal{U}(x)$ if $x \neq p$ and $\mathcal{U}'(p) = \{ U \cup B : U \subseteq \mathcal{U}(x), B \subseteq \mathcal{O} \}$. There is a topology $\mathcal{T}'$ on $X$ such that $\mathcal{U}'(x)$ is an open base at $x$ for each $x \in X$. It is clear that $\mathcal{T}'$ is strictly weaker than $\mathcal{T}$ (there is a $U \subseteq \mathcal{U}(p)$ which contains no set of $\mathcal{U}'(p)$ since $\mathcal{O}$ does not converge to $p$). Moreover, $\mathcal{T}'$ is certainly regular at each $x \neq p$, while regularity at $p$ follows readily from the fact that $\mathcal{O}$ is equivalent to a closed filter-base. Hence $\mathcal{T}$ is not minimal regular.

To establish the sufficiency of the condition, let $(X, \mathcal{T})$ be a regular space satisfying $(a)$ and let $\mathcal{T}'$ be a regular topology on $X$ which is weaker than $\mathcal{T}$. For arbitrary $x \in X$ let $\mathcal{U}(x)$ and $\mathcal{U}'(x)$ be the open neighborhood systems of $x$ in the $\mathcal{T}$ and $\mathcal{T}'$ topologies, respectively. The filter-base $\mathcal{U}'(x)$ is $\mathcal{T}'$-regular and has $x$ as its only adherent point. Since $\mathcal{T}'$ is weaker than $\mathcal{T}$, $\mathcal{U}'(x)$ is regular and has unique adherent point $x$ in $(X, \mathcal{T})$. By $(a)$ $\mathcal{U}'(x)$ converges to $x$ in $(X, \mathcal{T})$. Hence $\mathcal{U}(x)$ must be weaker than $\mathcal{U}'(x)$, and, since the reverse is true, it follows that $\mathcal{T}$ and $\mathcal{T}'$ are identical and that $\mathcal{T}$ is minimal regular.

**Remark.** The two previous results show that condition $(\beta)$ is necessary in order that a regular space be minimal regular. Whether it is sufficient is an open question. Theorem 3 below, however, throws some light on the problem.

**Lemma.** If the subspace $X$ of the regular space $Y$ satisfies $(\beta)$, then $X$ is closed in $Y$.

**Proof.** Suppose $p \in \overline{X} - X$. Let $\mathcal{U}$ and $\mathcal{V}$ be, respectively, the open and closed neighborhood systems of $p$ in $Y$. Then the filter-base $\mathcal{O} = \{ X \cap U : U \subseteq \mathcal{U} \}$ is open (relative to $X$), is equivalent to the closed (relative to $X$) filter-base $\{ X \cap V = V \subseteq \mathcal{V} \}$, and is therefore regular on $X$. As a filter-base on $Y$, $\mathcal{O}$ is stronger than $\mathcal{U}$ and hence has no adherent point other than $p$ in $Y$. It follows that $\mathcal{O}$ has no adherent point at all in $X$, a contradiction.

**Theorem 3.** Any completely regular space satisfying $(\beta)$ is compact and therefore minimal regular.

**Proof.** Let $X$ be completely regular and satisfy $(\beta)$ and let $Y$ be its Stone-Čech compactification. The above lemma yields the desired result.

**Theorem 4.** Any minimal regular subspace of a regular space is closed.

**Proof.** This is an immediate consequence of the lemma since the subspace must satisfy $(\beta)$. 


Remark. It is easy to see that a subspace of a minimal regular space which is both open and closed is itself minimal regular. The example of the next section shows that a closed subspace of a minimal regular space need not be minimal regular.

3. A minimal regular noncompact space. The example given here is a slight modification of an unpublished one due to Richard Arens of a regular space which is not completely regular. His example has also been used by Hewitt [3] in constructing a regular space on which every continuous real-valued function is constant.

Description of the space $(Z, 3)$. Let $J$ be the set of all integers, $\omega'$ the ordinals $\leq \omega$, and $\Omega'$ the ordinals $\leq \Omega$ (the first uncountable one). Equip each of these sets with the order topology and consider the space $J \times \omega' \times \Omega' \setminus \{(n, \omega, \Omega): n \in J\}$, the relative product topology being used. To obtain the space $Y$, make the following identifications and use the quotient topology $3^*$: for even $n$, identify $(n, \omega, \gamma)$ and $(n+1, \omega, \gamma)$; for odd $n$, identify $(n, x, \Omega)$ and $(n+1, x, \Omega)$. We will continue to use the symbols $(n, x, \gamma)$ for the points of $Y$, thus $(n, \omega, \gamma) = (n+1, \omega, \gamma)$ for even $n$. For $n \in J$, let $Q_n = \{(n, x, \gamma): x < \omega, \gamma < \Omega\}$ and $Z_n = Q_n = \{(n, x, \gamma): (x, \gamma) \neq (\omega, \Omega)\}$. Let $p$ and $q$ be points not in $Y$ and topologize $Z = \{p\} \cup \{q\} \cup Y$ by letting an open base at $p$ be all sets of the form

$$V_n(p) = \bigcup \left\{ Z_i: i > n \right\} \cup Q_n \cup \{p\}, \quad n = 1, 2, \ldots,$$

and an open base at $q$ be all sets of the form

$$V_n(q) = \bigcup \left\{ Z_i: i > n \right\} \cup Q_n \cup \{q\}, \quad n = 1, 2, \ldots,$$

which open bases at points of $Y$ are those they had in $3^*$. Let the resulting topology on $Z$ be $3$.

Properties of the space $(Z, 3)$. 1. $(Z, 3)$ is regular.

Proof. It is easy to see that singletons are closed, and regularity is clear except possibly at $p$ and $q$. Regularity at $p$, say, follows from $[V_{n+1}(p)]^{-} \subseteq V_n(p)$.

We will say that a set $S \subseteq Z$ gets into the $n$-corner if whenever $x_0 < \omega, y_0 < \Omega$, there is a point $(n, x, \gamma) \in S$ for some $x > x_0$ and $y > y_0$.

2. If the open set $U$ gets into the $n$-corner, then there is an infinite sequence $\{x_i\}$ of distinct finite ordinals such that $(n, x_i, \Omega) \in U$.

Proof. If not, there is an $x_0 < \omega$ such that if $x_0 < x < \omega$, $(n, x, \Omega) \not\in U$ and hence there is a $y_x < \Omega$ such that $(n, x, y) \in U$ for $y_x < y$. Since $\{y_x: x_0 < x < \omega\}$ is countable, its least upper bound, $y_0$, is less than $\Omega$. Therefore if $x_0 < x < \omega$ and $y_0 < y$, then $(n, x, y) \not\in U$. Since $U$ gets into the $n$-corner, it must then be that $(n, \omega, \gamma) \in U$ for some $\gamma > y_0$. But
since $U$ is open, there is an $x$, $x_0<x<\omega$, such that $(n, x, y) \in U$. This contradiction establishes the property.

3. Let $U$, $V$, and $W$ be open sets such that $U \subset \overline{U} \subset V \subset \overline{V} \subset W$. Then if $U$ gets into the $n$-corner, $W$ gets into the $(n-1)$- and $(n+1)$-corners.

**Proof for $n$ odd.** (The proof for the case $n$ even is similar.) Take $x_0<\omega$, $y_0<\omega$. By property 2, there are infinitely many distinct $x_t$ such that $x_0< x_t<\omega$ and $(n, x_t, \Omega) \in \overline{U}$. Since $(n+1, x_t, \Omega) = (n, x_t, \Omega) \in \overline{U} \subset W$, $W$ gets into the $(n+1)$-corner. Since $(n, x_t, \Omega) \in V$, there exists, for each $i$, a $y_i<\omega$ such that if $y>y_i$, then $(n, x_t, y) \in V$. Let $y'$ be the least upper bound of the set $\{y_0, y_1, y_2, \cdots \}$. Then for any $y$, $y'<y<\omega$, $(n, x_t, y) \in V$ for all $x_t$; hence $(n, \omega, y) = (n-1, \omega, y) \in \overline{V} \subset W$, and $W$ gets into the $(n-1)$-corner.

4. If $\mathfrak{B}$ is a regular filter-base and, for some $n$, each set of $\mathfrak{B}$ gets into the $n$-corner, then $p$ and $q$ are adherent points of $\mathfrak{B}$. 

**Proof.** Let $N$ be a neighborhood of $p$ and $B \in \mathfrak{B}$. There is an integer $k$ such that $Q_k \subset V\alpha(p) \subset N$; let $h = k - n$. Since $\mathfrak{B}$ is regular, there are $2h+1$ sets $U_i \in \mathfrak{B}$ such that $U_1 \subset \underline{U}_1 \subset U_2 \subset \cdots \subset U_{2n+1} = B$. 

Since $U_1$ gets into the $n$-corner, $h$ applications of property 3 shows that $B = U_{2n+1}$ gets into the $(n+h)\alpha$-corner; i.e., $B \cap Q_k \neq \emptyset$, whence $B \cap N \neq \emptyset$, and $p$ is an adherent point of $\mathfrak{B}$. The case for $q$ is similar.

5. $(Z, 3)$ is not completely regular and hence not compact.

**Proof.** Let $f$ be a bounded, real-valued continuous function on $Z$. For some fixed $n$ and each $y<\omega$, let $g(y) = f(n, \omega, y)$. Then $g$ is continuous, and it is well-known (e.g., [4, p. 167, ex. Q]) that there is a $y_0<\omega$ and a constant $c$ such that $g(y) = c$ for $y>y_0$. It follows that each set of the regular filter-base $\{ \{ p \in Z : |f(p) - c| < \epsilon \} : \epsilon > 0 \}$ gets into the $n$-corner. Since, by property 4, $p$ and $q$ are adherent points of this filter-base, it is clear that $f(p) = f(q) = c$ and $(Z, 3)$ is not completely regular.

In the proof of the following property we repeatedly use the elementary fact that if $\mathfrak{B}$ is a regular filter-base and $C \in \mathfrak{B}$, then $C = \bigcap B \in \mathfrak{B}$ is a regular filter-base equivalent to $\mathfrak{B}$. We will call $C$ the $C$-section of $\mathfrak{B}$.

6. $(Z, 3)$ is minimal regular.

**Proof.** Let $\mathfrak{B}$ be a regular filter-base with unique adherent point $r$. We will show that $\mathfrak{B}$ converges to $r$; the property will then follow from Theorem 2.

**Case 1.** $r \neq p, q$. Then some set $C \in \mathfrak{B}$ meets only a finite number of $Z_a$'s. Let $C$ be the $C$-section of $\mathfrak{B}$; then there is an integer $k$ such that
each set of $C$ is a subset of $K = \bigcup \{z_n: |n| \leq k\}$. It follows from property 4 that for each $n$, $|n| \leq k$, there is a set $D_n \subseteq C$ which does not get into the $n$-corner. Let $D$ be a set of $C$ lying in $\bigcap \{D_n: |n| \leq k\}$; then ordinals $x_0 < \omega$, $y_0 < \Omega$ exist such that $D$ does not meet the open set $W = \{(n, x, y): x > x_0, y > y_0\}$. Hence $D$, the $D$-section of $C$, is a filter-base equivalent to $\emptyset$, and each of its sets lies in the compact subspace $K - W$ of $Z$. It is clear that $\emptyset$, and hence $\mathcal{B}$, must converge to their unique adherent point $r$.

Case 2. $r = p$. (The proof for the case $r = q$ is similar.) If $\mathcal{B}$ does not converge to $p$, there is a neighborhood $V_k(p)$ which contains no set of $\mathcal{B}$. Since $q$ is not an adherent point of $\mathcal{B}$, there is an integer $h$ and a set $C$ of $\mathcal{B}$ such that $C \cap Z_n = \emptyset$ for $n < h$. It follows from property 4 that for each $n$, $h \leq n \leq k$, there is a set $D_n$ in the $C$-section $\mathcal{E}$ of $\mathcal{B}$ which does not get into the $n$-corner. Let $D$ be a set of $C$ lying in $\bigcap \{D_n: h \leq n \leq k\}$; then ordinals $x_0 < \omega$ and $y_0 < \Omega$ exist such that $D$ does not meet the set $W = \{(n, x, y): h \leq n \leq k, x > x_0, y > y_0\}$. The $D$-section $\mathcal{E}$ of $C$ is a filter-base equivalent to $\emptyset$ and each of its sets meets the compact set $F = \bigcup \{Z_n: h \leq n \leq k\} - W$. Hence $\mathcal{E} = \{F \cap E: E \in \mathcal{D}\}$ is a filter-base stronger than $\emptyset$ and each of its sets is contained in $F$. Since $F$ is compact, $\mathcal{E}$, and hence $\mathcal{B}$, must have an adherent point $z \in F$. Since $z \neq p$, a contradiction results.

7. $(Z, \mathcal{S})$ has a closed subspace which is not minimal regular.

Proof. Let $S = \{(1, x, \Omega): x < \omega\}$. It is clear that $S$ is a closed subset of $Z$. But, with the relative topology, $S$ is an infinite discrete space, which is certainly not minimal regular.

References


