THE RAYLEIGH FUNCTION

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The present paper is a study of a set of symmetric functions of
the zeros of \( J_\nu(z) \), the Bessel function of the first kind. Let the zeros
of \( z^{-\nu} J_\nu(z) \) be denoted by \( j_{\nu,m}, \quad m = 1, 2, \cdots \), where
\[ |R(j_{\nu,m})| \leq |R(j_{\nu,m+1})| \]. And let

\[
\sigma_{2n}(\nu) = \sum_{m=1}^{\infty} (j_{\nu,m})^{-2n}, \quad n = 1, 2, \cdots
\]

We propose to call \( \sigma_{2n}(\nu) \) the Rayleigh function of order \( 2n \). The
Rayleigh functions of odd orders are identically zero.

These functions were first used by Euler to determine the three
smallest zeros of \( J_0(2\sqrt{z}) \); and Rayleigh, independently, calculated
the smallest positive zero of \( J_\nu(z) \) with the aid of these functions. The
Rayleigh functions have been the subject of a number of investiga-
tions by Cayley, Graeffe, Graf and Gubler, Watson, Kapteyn,
Forsyth and others [6, p. 502].

In this paper we shall understand by the Bernoulli and Genocchi
numbers, the entities \( B_n \) and \( G_n \) defined as follows:

\[
B_n = \sum_{k=0}^{n} \binom{n}{k} B_k, \quad n \neq 1,
\]

\[
B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = 1, \quad B_3 = 0, \quad B_4 = -\frac{1}{4}, \cdots
\]

\[
G_n = 2(1 - 2^n)B_n;
\]

(see [4; 3]).

From the well-known formula

\[
z^{-1/2} J_{1/2}(z) = \frac{\sqrt{2}}{\pi} \sin z/z,
\]

(see [6, p. 54]), the roots of \( z^{-1/2} J_{1/2}(z) \) are seen to be \( k\pi, \quad k = 1, 2, \cdots \).

Hence

\[
\sigma_{2n}\left(\frac{1}{2}\right) = \sum_{k=1}^{\infty} (k\pi)^{-2n} = \pi^{-2n} \zeta(2n) = (-1)^{n-1} \frac{2^{2n-1}}{(2n)!} B_{2n}.
\]

Similarly, since \( z^{1/2} J_{-1/2}(z) = \sqrt{2/\pi} \cos z \), the roots of \( z^{1/2} J_{-1/2}(z) \) are

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527
\[(k - \frac{1}{2})\pi, \; k = 1, 2, \cdots.\] Therefore,
\[
\sigma_{2n}\left(-\frac{1}{2}\right) = \pi^{-2n} \sum_{k=1}^{\infty} \left(k - \frac{1}{2}\right)^{-2n} = (2^{2n} - 1)\pi^{-2n}G_{2n},
\]
(5)
\[
= (-1)^n \frac{2^{2n-2}}{(2n)!} G_{2n}.
\]

A generating function for \(\sigma_{2n}(\nu)\) can be obtained. Let \(J_\nu(z)\) be expressed as a Weierstrassian product
\[
J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \prod_{n=1}^{\infty} \left\{1 - \frac{z^2}{\nu^2(n)}\right\},
\]
(see [6, p. 498]). Differentiating logarithmically with respect to \(z\),
\[
J_\nu'(z)/J_\nu(z) = \nu/z + \sum_{k=1}^{\infty} \frac{(-2z)/(\nu^2 - z^2)}{J_{\nu,k}^2}
= 1/z \left\{\nu - 2\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{z^2}{\nu^2(J_{\nu,k})}\right\},
\]
\[
zJ_\nu'(z)/J_\nu(z) = \nu - 2\sum_{n=1}^{\infty} \sigma_{2n}(\nu)z^{2n},
\]
(6)
\[
\frac{1}{2} zJ_{\nu+1}(z)/J_\nu(z) = \sum_{n=1}^{\infty} \sigma_{2n}(\nu)z^{2n}.
\]
Substituting \(z = 1\) in (6) yields
\[
\sum_{n=1}^{\infty} \sigma_{2n}(\nu) = \frac{1}{2} J_{\nu+1}(1)/J_\nu(1);
\]
(7)
\[
\sum_{n=1}^{\infty} \sigma_{2n}(\nu + k) = \frac{1}{2} J_{\nu+k+1}(1)/J_{\nu+k}(1), \quad k = 0, 1, 2, \cdots.
\]
(8)
Setting \(\nu = +\frac{1}{2}\) and \(-\frac{1}{2}\) in (6), the following are obtained in view of (4) and (5),
\[
\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}}{(2n)!} B_{2n}z^{2n} = 1 - z \cot z,
\]
(9)
\[
\sum_{n=1}^{\infty} (-1)^{n} \frac{2^{2n-1}}{(2n)!} G_{2n}z^{2n} = z \tan z.
\]
(10)
Taking the continued fraction representation of the ratio of two
Bessel functions,
\[ J_{r+1}(z)/J_r(z) = \frac{z}{2(v + 1)} - \frac{z^2}{2(v + 1)} - \cdots , \]
(see [6, p. 153]) then substituting \( z = 1 \), and using (7),
\[ \sum_{n=1}^{\infty} \sigma_{2n}(v) = \frac{1}{2} \cdot \frac{1}{2(v + 1)} - \frac{1}{2(v + 2)} - \cdots . \]

Set \( v = \pm \frac{1}{2} \) in (11), then
\[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}}{(2n)!} B_{2n} = \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \cdots = 1 - \cot (1) = 0.3579, \]
\[ \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n-1}}{(2n)!} G_{2n} = \frac{1}{1} - \frac{1}{3} - \frac{1}{5} - \cdots = \tan (1) = 1.5574. \]

A recurrence formula for the functions \( \sigma_{2n}(v) \) may be derived from (6). Let (6), be written as
\[ \frac{1}{2} z J_{r+1}(z) = J_r(z) \sum_{n=1}^{\infty} \sigma_{2n}(v) z^{2n}. \]
Substituting the series for \( J_r(z) \) and \( J_{r+1}(z) \), and identifying the coefficients of \( z^{2n} \) on both sides,
\[ \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!(n-k)!} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} n + k \\ k \end{array} \right) \sigma_{2k}(v) = n. \]
Substitute \( v = \pm \frac{1}{2} \) in (14), then in view of (4) and (5),
\[ \sum_{k=1}^{n} 2^{2k-1} \left( \begin{array}{c} 2n + 1 \\ k \end{array} \right) B_{2k} = n, \]
(see [5, p. 174])
\[ - \sum_{k=1}^{n} 2^{2k-2} \left( \begin{array}{c} 2n \\ 2k \end{array} \right) G_{2k} = n. \]
which are well-known Bernoulli and Genocchi recurrence formulas.

A determinant representation of \( \sigma_{2n}(\nu) \) may be given. If in (14) the upper limit of the summation is taken as 1, 2, \( \cdots \), \( n \), then \( n \) linear equations involving \( n \) functions \( \sigma_2(\nu), \sigma_3(\nu), \cdots, \sigma_{2n}(\nu) \) are obtained. The determinant \( \Delta \) of this system is triangular. Hence, the value of \( \Delta \) is equal to the product of its elements on the main diagonal,

\[
\Delta = (-1)^{n(n-1)/2} \cdot 2^{n(n+1)} \cdot n! \prod_{k=1}^{n} (\nu + k)^{n-k+1}.
\]

Then by Cramer's rule,

\[
\sigma_{2n}(\nu) = (-1)^{n-1} \frac{n-1}{n!} \prod_{k=1}^{n} (\nu + k)^{n-k-1} \cdot D,
\]

where,

\[
D =
\begin{vmatrix}
(1)_{(\nu + 1)} & 0 & 0 & \cdots & 0 & 1 \\
(1)_{(\nu + 2)} & (2)_{(\nu + 2)} & 0 & \cdots & 0 & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(1)_{(\nu + n)} & (2)_{(\nu + n)} & (3)_{(\nu + n)} & \cdots & (n-1)_{(\nu + n)} & n
\end{vmatrix}.
\]

Substitution of \( \nu = \pm \frac{1}{2} \) in (17) yields, respectively,

\[
B_{2n} = \frac{n!}{(2n+1)!} 2^{-n+1} D_1,
\]

\[
G_{2n} = -2^{-2n+2} D_2,
\]

where

\[
D_1 =
\begin{vmatrix}
(3) & 0 & 0 & \cdots & 0 & 1 \\
(5) & (5) & 0 & \cdots & 0 & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(2n+1) & \cdots & (2n+1) & \cdots & (2n+1) & n
\end{vmatrix}.
\]
\[
D_2 = \begin{pmatrix}
2 & 0 & 0 & \cdots & 0 & 1 \\
4 & 2 & 0 & \cdots & 0 & 2 \\
4 & 4 & 2 & \cdots & 0 & 2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
2n & \cdots & 2n & 2n & \cdots & 2n \\
2 & \cdots & 2 & \cdots & \cdots & 2
\end{pmatrix}.
\]

Other formulas for \(\sigma_{2n}(\nu)\) may be derived. Let (6) be written as
\[
\{zJ_{\nu+1}(z)/J_{\nu+1}(z)\} \{zJ_{\nu+1}(z)/J_{\nu}(z)\}
= 4 \left\{ \sum_{k=1}^{\infty} \sigma_{2k}(\nu + 1)z^{2k} \right\} \left\{ \sum_{k=1}^{\infty} \sigma_{2k}(\nu)z^{2k} \right\}.
\]

Then in view of the well-known formula,
\[
J_{\nu+1}(z) = \frac{2(\nu + 1)}{z} J_{\nu+1}(z) - J_\nu(z),
\]
(see [6, p. 45]), the above becomes
\[
-\frac{1}{4} z^2 + \frac{1}{2} (\nu + 1)zJ_{\nu+1}(z)/J_\nu(z) = \left\{ \sum_{k=1}^{\infty} \sigma_{2k}(\nu + 1)z^{2k} \right\} \left\{ \sum_{k=1}^{\infty} \sigma_{2k}(\nu)z^{2k} \right\}.
\]

Identifying the coefficients of \(z^{2n}\) on both sides
\[
(\nu + 1)\sigma_{2n}(\nu) = \sum_{k=1}^{n-1} \sigma_{2k}(\nu + 1)\sigma_{2n-2k}(\nu).
\]

Substitution of \(\nu = -\frac{1}{2}\), in (20) yields
\[
G_{2n} = -\sum_{k=1}^{n-1} \binom{2n}{2k} B_{2k} G_{2n-2k}.
\]

Consider the well-known formulas
\[
\frac{d}{dz} (z^{-\nu}J_\nu(z)) = -z^{-\nu}J_{\nu+1}(z),
\]
(see [6, p. 45]),
\[
\frac{d}{dz} (z^{\nu+1}J_{\nu+1}(z)) = z^{\nu+1}J_\nu(z).
\]

It follows that
\[
\frac{d}{dz} \left( \frac{z^{r+1} J_{r+1}(z)}{z^{-r} J_{r}(z)} \right) = z \frac{2r+1 (1 + J_{r+1}^2(z))}{J_{r+1}^2(z)}
\]

That is,
\[
\left( \frac{z}{2} \frac{J_{r+1}(z)}{J_{r}(z)} \right)^2 = -z^2 \frac{1}{4} + \frac{1}{2} z^{-2r+1} \frac{d}{dz} \left( \frac{z}{2} \frac{J_{r+1}(z)}{J_{r}(z)} z^{2r} \right).
\]

Substituting (6) in this relation and identifying the coefficients of \(z^{2n}\),
\[
(22) \quad (\nu + n) \sigma_{2n}(\nu) = \sum_{k=1}^{\infty} \sigma_{2k}(\nu) \sigma_{2n-2k}(\nu).
\]

Substitute \(\nu = \pm \frac{1}{2}\) in (22), then
\[
(23) \quad -(1 + 2n) B_{2n} = \sum_{k=1}^{n-1} \left( \begin{array}{c} 2n \\ 2k \end{array} \right) B_{2k} B_{2n-2k},
\]

(see [5, p. 66])
\[
(24) \quad -2(1 - 2n) G_{2n} = \sum_{k=1}^{n-1} \left( \begin{array}{c} 2n \\ 2k \end{array} \right) G_{2k} G_{2n-2k}.
\]

Let (6) be multiplied by itself,
\[
J_{r+1}^2(z) = 4 J_{r}^2(z) \left( \sum_{k=1}^{\infty} \sigma_{2k}(\nu) z^{2k-1} \right)^2,
\]

then substituting the well-known series for \(J_{r+1}^2(z)\) and \(J_{r}^2(z)\), viz.,
\[
J_{r+1}^2(z) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(2\nu + 3 + 2k)}{k! \Gamma(2\nu + 3 + k) \Gamma^2(\nu + 2 + k)} \left( \frac{z}{2} \right)^{2r+2+2k},
\]
\[
J_{r}^2(z) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(2\nu + 1 + 2k)}{k! \Gamma(2\nu + 1 + k) \Gamma^2(\nu + 1 + k)} \left( \frac{z}{2} \right)^{2r+2k},
\]

(see [6, p. 147]), and identifying the coefficients of \(z^{2n}\) on both sides, the following is obtained
\[
\sum_{k=1}^{n} (-1)^{k-1} \left( \begin{array}{c} 2\nu + 2n - 2k \\ n - k \end{array} \right) (k!)^2 \left( \begin{array}{c} \nu + n \\ k \end{array} \right) z^{2r+1} \sum_{s=1}^{k} \sigma_{2s}(\nu) \sigma_{2(k-s-1)}(\nu) = \left( \begin{array}{c} 2\nu + 2n \\ n - 1 \end{array} \right)
\]

(25)

In view of (22) this reduces to
\[
\sum_{k=1}^{n} (-1)^{k-1} 4k \binom{2n + 2k - 2n}{n-k} \left( \frac{(k-1)!}{(k+1)!} \right)^2 \left( \frac{n}{k} \right)^2 \right) \sigma_{2k}(\nu)
\]

\[
= 2v + 1 \left( \begin{array}{c} 2v + 2n - 2 \\ n + 1 \end{array} \right) 
\]

Substitute \(v = \pm \frac{1}{2}\) in (26), then

\[
\sum_{k=1}^{n-1} (1 + 2k) \binom{2n}{2k} B_{2k} = 2n - 1, 
\]

\[
\sum_{k=1}^{n-1} (1 - 2k) \binom{2n}{2k} G_{2k} = 2(2n - 1)G_{2n}. 
\]

The Rayleigh functions of odd order are zero,

\[
\sigma_1(\nu) = 0, \quad \sigma_3(\nu) = 0, \quad \sigma_5(\nu) = 0, \ldots .
\]

Identifying the coefficients of \(\nu^2\) in (6), an explicit expression for \(\sigma_2(\nu)\) is obtained. Then (20) may be used to get explicit expressions for the Rayleigh functions of higher orders. Thus,

\[
\frac{1}{2(\nu + 1)}; 
\]

\[
\frac{1}{2(\nu + 1)^2(\nu + 2)}; 
\]

\[
\frac{2}{2(\nu + 1)^3(\nu + 2)(\nu + 3)}; 
\]

\[
\frac{5\nu + 11}{2(\nu + 1)^4(\nu + 2)^2(\nu + 3)(\nu + 4)}. 
\]

The first twelve Rayleigh functions are given in [1, pp. 405-407].

**BIBLIOGRAPHY**


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