ON THE EXTENSION OF MODULAR MAXIMAL IDEALS

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1. Introduction. Let $B$ be a complex commutative Banach algebra and let $A$ be a subalgebra of $B$. In [1, §23] and [3] Silov, about twenty years ago, investigated conditions under which some or all of the modular maximal ideals of $A$ are contained in modular maximal ideals of $B$. We re-examine this question for noncommutative Banach algebras. Here neither $A$ nor $B$ need be commutative and the ideals in question are the modular maximal two-sided ideals of $A$ and $B$. Simple examples show that, even if $A$ is a commutative subalgebra of $B$ and Silov’s conditions are fulfilled, no m.m. ideal of $A$ need be contained in a m.m. ideal of $B$. Success can be hoped for only if $A$ is favorably situated in $B$.

Suppose (for simplicity) that $A$ is closed in $B$. Consider the set $\mathcal{O}$ of all m.m. ideals $N$ of $B$ such that $xy - yx \in N$ for all $x \in A$, $y \in B$. Each $N \in \mathcal{O}$ determines a multiplicative linear functional $x \mapsto x(N)$ on $A$. Let $\mathcal{J}$ be the set of m.m. ideals of $A$ which are the null spaces of multiplicative linear functionals on $A$. The algebra $A$ can be represented homomorphically as an algebra of continuous functions on $\mathcal{J}$ in the fashion of Gelfand. Let $\Delta(A)$ be the Silov boundary of $A$ in $\mathcal{J}$. Then each $M \in \Delta(A)$ is contained in a m.m. ideal of $B$ if $\sup |x(N)| = \nu(x)$ for all $x \in A$ where the sup is taken over $\mathcal{O}$ and $\nu(x)$ is the spectral radius of $x$. A sufficient condition for this relation to hold on $A$ is that $xy - yx$ lie in the radical $J$ of $B$ for all $x \in A$, $y \in B$. An example shows that this can take place where $A$ properly contains $C + J$ where $C$ is the center of $B$.

2. Algebraic preliminaries. Let $B$ be a ring with a subring $A$ and denote by $\mathcal{M}(B)$ the set of all m.m. ideals of $A(B)$. Let $I$ be a two-sided ideal of $B$ and set $C(I) = \{x \in B| xy - yx \in I \text{ for all } y \in B\}$. It is readily seen that $C(I)$ is a subring of $B$. We assume that $C(I) \supseteq A$. Given $M \in \mathcal{M}$ we let $\alpha(M; I)$ denote the set of all finite sums of the
form $\sum x_k z_k + u$ where each $x_k \in M$, $z_k \in B$ and $u \in I$. We set $\beta(M; I) = \{w \in B \mid wy \in \alpha(M; I) \forall y \in A\}$. Since $C(I) \supset A$ it is clear that $\alpha(M; I)$ can be described as the set of all sums $\sum x_k z_k + u$ so that $\alpha(M; I)$ is a two-sided ideal of $B$. Also, $\beta(M; I) = \{w \in B \mid wy \in \alpha(M; I) \forall y \in A\}$ so that $\beta(M; I)$ is a two-sided ideal of $B$ containing $M$ and $I$.

We let $J$ denote the radical of $B$ and $S$ its strong radical (the intersection of its m.m. ideals), see [2, p. 59]. Note that $S \supset J$.

2.1. Lemma. If $\beta(M; I) \neq B$ there exists $N \in \mathfrak{N}$ such that $N \cap A = M$. If $I$ is the strong radical $S$ of $B$ and there exists $N \in \mathfrak{N}$ such that $N \cap A = M$ then $\beta(M; S) \neq B$.

Suppose that $\beta(M; I) \neq B$ and let $j$ be an identity for $A$ modulo $M$. Take $z \in B$ and $y \in A$. We can write $zy = yz + v$ where $v \in I$. Then $(jz - z)y = (jy - y)z + jv - v \in \alpha(M; I)$. Therefore $jz - z \in \beta(M; I)$ and likewise $zj - z$ for all $z \in B$. Thus $\beta(M; I)$ is a proper modular two-sided ideal of $B$ containing $M$ and is therefore contained in some $AG9I$. Since $j \in N$ we see that also $N \cap A = M$.

Next take the case $I = S$. Suppose $N \in \mathfrak{N}$ and $N \cap A = M$. Since $N \supset S$ we also see that $N \supset \alpha(M; S)$. If $\beta(M; S) = B$ then $j^2 \in \alpha(M; S) \subset N$. Since $j^2 - j \in M \subset N$ it follows that $j \in N \cap A = M$ which is impossible.

The following example shows that, even for Banach algebras $C(J)$ can be larger than $C+J$ where $C$ is the center of $B$. Let $B$ be the Banach space of all complex-valued continuous functions on $[0, 1]$ made into a Banach algebra by defining products by the rule $fg(t) = f(t)g(t)$, $0 \leq t \leq 1$. For this algebra, $J = S = \{f \in B \mid f(0) = 0\}$, $C = (0)$ and $C(J) = B$.

3. Extension of maximal ideals. We adopt the notation of §2 except that $A$ and $B$ are complex Banach algebras (with $A$ algebraically embedded in $B$). Let $\mathfrak{Q} = \{N \in \mathfrak{N} \mid xy - yx \in N \forall x \in A, y \in B\}$.

For each $N \in \mathfrak{N}$ let $x \rightarrow \pi_N(x)$ be the natural homomorphism of $B$ onto $B/N$. The latter is a simple Banach algebra with an identity. If $N \in \mathfrak{Q}$, the image of $A$ in $B/N$ lies in the center of $B/N$. But that center is a field and so, by the Gelfand-Mazur theorem, is the set of scalar multiples of its identity $\pi_N(v)$ (see [2, p. 85]). If we write $\pi_N(x) = x(N)\pi_N(v)$ where $x(N)$ is a scalar, the mapping $x \rightarrow x(N)$ is a multiplicative linear functional on $A$ (trivial if $N \supset A$). Set

$$\beta(x) = \sup_{N \in \mathfrak{Q}} |x(N)|, \quad x \in A.$$

Let $\mathfrak{B}$ denote the subset of $\mathfrak{M}$ consisting of all zero sets of multi-
plicative linear functionals on $A$. For each $M \in \mathfrak{B}$ denote the corresponding functional by $x(M)$. Using the Gelfand theory we can represent $A$ homomorphically as an algebra of continuous functions vanishing at infinity on $\mathfrak{B}$ where we give to $\mathfrak{B}$ its weak topology defined by the functions $x(M)$, $x \in A$. We may then speak of the Silov boundary $\Delta(A)$ of $A$ in $\mathfrak{B}$ in the usual way [2, p. 132]. Set

$$\alpha(x) = \sup_{M \in \mathfrak{B}} |x(M)|, \quad x \in A.$$ 

It is clear that

$$\beta(x) \leq \alpha(x), \quad x \in A.$$

For each $x \in A$ let $\|x\|_0(\|x\|)$ be its norm as an element of the Banach algebra $A(B)$. Consider the spectral radii $\nu_A(x) = \lim \|x^n\|_0^{1/n}$ and $\nu_B(x) = \lim \|x^n\|_0^{1/n}$ of $x$ computed for $A$ and $B$ respectively. The relation $\nu_B(x) \leq \nu_A(x)$ is automatic. If $A$ is a closed subalgebra of $B$, $\nu_B(x) = \nu_A(x)$ for $x \in A$. We shall also have occasion to consider the spectrum of $x$ computed in $A(B)$ which we denote by $\text{sp}(x|A)$ ($\text{sp}(x|B)$).

In the notation above, for $x \in A$ we have, for each positive integer $m$,

$$|x(M)| \|x_N(\cdot)\|^{1/m} = \|x_N(x^m)\|^{1/m} \leq \|x^m\|^{1/m}.$$

Letting $m \to \infty$ we see that

$$\beta(x) \leq \nu_B(x), \quad \alpha(x) \leq \nu_A(x), \quad x \in A.$$

Let $E(\mathfrak{Q})$ denote the set of $M \in \mathfrak{M}$ for which there exists $N \in \mathfrak{Q}$ such that $N \cap A = M$.

3.1. Theorem. The following statements are equivalent.
(a) $\nu_B(x) = \nu_A(x)$ for all $x \in A$.
(b) $E(\mathfrak{Q}) \supseteq \Delta(A)$ and $\alpha(x) = \nu_A(x)$, $x \in A$.

Suppose (a) holds. It follows from (1) and (2) that $\alpha(x) = \beta(x)$, $x \in A$. Let $k(\mathfrak{Q})$ denote the intersection of the $N \in \mathfrak{Q}$ and let $M_0 \in \Delta(A)$. By Lemma 2.1 it is sufficient to show that $\beta(M_0; k(\mathfrak{Q})) \neq B$ inasmuch as any $N \in \mathfrak{M}$ such that $N \supseteq \beta(M_0; k(\mathfrak{Q}))$ contains $k(\mathfrak{Q})$ and so lies in $\mathfrak{Q}$. Suppose the contrary. We can then write (where $j$ is an identity for $A$ modulo $M_0$)

$$j^2 = \sum_{k=1}^n x_k s_k + u,$$

where each $x_k \in M_0$, $s_k \in B$ and $u \in k(\mathfrak{Q})$. 

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For each $N \in \mathcal{Q}$ consider the identity $\pi_N(v)$ of $B/N$. In the quotient algebra norm, $\|\pi_N(v)\| \geq 1$. An equivalent norm $\|\pi_N(x)\|_1$ may be introduced into $B/N$ so that $\|\pi_N(v)\|_1 = 1$. We suppose that this procedure has been followed for each $N \in \mathcal{Q}$ and set

$$\|w\| = \sup_{N \in \mathcal{Q}} \|\pi_N(w)\|_1, \quad w \in B.$$  

Note that $\|w\|$ is defined on all of $B$ whereas $\alpha(x)$ and $\beta(x)$ are only defined on $A$. It is easy to see that

$$\|x\| = \beta(x), \quad x \in A.$$  

Our argument is now an adaptation of one of Šilov [1, §23]. Without loss of generality we may assume that, in (3), $\beta(x_k) \leq 1$ for $k = 1, \ldots, n$. Let $a$ be any positive number, $a > \max ||x_k||$ and $2na > 1$. Note that $j(M_0) = 1$ and $x_k(M_0) = 0$, $k = 1, \ldots, n$. Consider the neighborhood $\mathcal{B}$ of $M_0$ in $\mathcal{B}$ defined by the inequalities

$$\begin{align*}
\{ M \in \mathcal{B} & \mid |x_k(M)| < 1/(2na), \ k = 1, \ldots, n \} \\
\{ M \in \mathcal{B} & \mid |j^2(M) - 1| < 1/3 \}.
\end{align*}$$

From [2, p. 135] there exists $y \in A$ such that

$$\sup_{M \in \mathcal{B}} |y(M)| = 1, \quad \sup_{M \in \mathcal{B}} |y(M)| < 1/(2na).$$

Now, for $M \in \mathcal{B}$, we have $|j^2y(M)| = |j^2(M)| |y(M)| \geq 2 |y(M)|/3$ by (6) so that

$$\beta(j^2y) \geq 2/3.$$  

Since $z_ky - yz_k \in k(\mathcal{Q})$ we can from (3) write

$$j^2y = \sum_{k=1}^n (z_ky)z_k + w$$

where $w \in k(\mathcal{Q})$. Note that $\pi_N(w) = 0$ for each $N \in \mathcal{Q}$. Therefore we get from (4), (5) and (9) that

$$\beta(j^2y) \leq \sum_{k=1}^n \|x_ky\| \|z_k\||.$$  

Next, for $M \in \mathcal{B}$, $|x_ky(M)| \leq |x_k(M)| < 1/(2na)$ while for $M \in \mathcal{B}$, $|x_ky(M)| \leq \alpha(x_k) |y(M)| \leq |y(M)| < 1/(2na)$ so that, by (5), $\|x_ky\| \leq 1/(2na)$. From (10) we see that $\beta(j^2y) \leq 1/2$. This is contrary to (8) and we have shown that (a) implies (b).

Suppose (b) and let $x \in A$. By the definition of the Šilov boundary
there exists $M_0 \in \Delta(A)$ such that $|x(M_0)| = v_A(x)$. Let $j$ be an identity for $A$ modulo $M_0$ and let $N \in \mathfrak{N}$ have the property that $N \cap A = M_0$. As $j^2 - j \in M_0 \subset N$ and $j(N) \neq 0$ we see that $j(N) = 1$. Suppose $x(M_0) = a$. Then $x - aj \in N$ whence $|x(N)| = v_A(x)$. Therefore $\beta(x) \geq v_A(x)$ and, by (2), we see that $v_A(x) = v_B(x) = \beta(x)$.

3.2. Lemma. If $C(J) \supset A$ then $\mathfrak{Q} = \mathfrak{N}$ and $\beta(x) = v_B(x)$ for all $x \in A$.

Since $J \subseteq S$ it follows from $C(J) \supset A$ that $\mathfrak{Q} = \mathfrak{N}$. Let $x \in A$ and let $a \in \text{sp}(x|B)$, $a \neq 0$. Set $K = \{a^{-1}xy - y|y \in B\} + J$. Clearly $K = \{a^{-1}yx - y|y \in B\} + J$ and is a modular two-sided ideal of $B$. Suppose that $K = B$. Then there exists $y \in B$, $z \in J$ such that $a^{-1}x + y - a^{-1}xy = z$. This implies that $a^{-1}x$ is right quasi-regular in $B$. Likewise $a^{-1}x$ is left quasi-regular and hence quasi-regular in $B$ which is impossible. Thus there exists $N \in \mathfrak{N}$ with $K \subset N$. Since $J \subseteq N$ it follows that $\pi_N(a^{-1}x)$ is the identity of $B/N$. Therefore $x(N) = a$. Conversely, if $x(N) = a \neq 0$ for some $N \in \mathfrak{N}$ then $\{a^{-1}xy - y|y \in B\}$ lies in $N$ so that $a \in \text{sp}(x|B)$. Thus $\beta(x) = v_B(x)$.

Lemma 3.2 gives a sufficient condition for the applicability of Theorem 3.1. It follows immediately that if $A$ is a closed subalgebra of the center $C$, each $M \in \Delta(A)$ is contained in some $N \in \mathfrak{N}$.

We next consider the strong structure space $\mathfrak{M}(\mathfrak{R})$ of $A(B)$ in its hull-kernel topology [2, p. 78]. For the notion of a completely regular Banach algebra see [2, p. 83]. We can be more specific than in Theorem 3.1 if $A$ is completely regular.

3.3. Theorem. Suppose that $A$ is completely regular and that $v_A(x) = v_B(x) = \beta(x)$ for all $x \in A$. Then $E(\mathfrak{Q}) = \mathfrak{N}$.

Let an identity be adjoined to $B$ forming $B_1$ and let $A_1$ denote the corresponding augmentation of $A$. Let $\mathfrak{M}_1(\mathfrak{R}_1)$ be the strong structure space of $A_1(B_1)$ and let $\mathfrak{Q}_1 = \{N_1 \in \mathfrak{N}_1|wv - vw \in N_1$ for all $w \in A_1$, $v \in B_1\}$. If we write $w = \lambda + x$, $v = \mu + y$ where $\lambda$, $\mu$ are scalars, $x \in A$ and $y \in B$ we see that $N_1 \in \mathfrak{Q}_1$ if and only if $xy - yx \in N_1$ for all $x \in A$, $y \in B$. The mapping $N_1 \rightarrow N_1 \cap B$ takes $\mathfrak{M}_1 \sim \{B\}$ onto $\mathfrak{N}_1$. We then have $N_1 \in \mathfrak{Q}_1$, $N_1 \neq B$ if and only if $N_1 \cap B \in \mathfrak{Q}$ and all elements of $\mathfrak{Q}$ are obtainable in this way. Next we note that $\mathfrak{R}_1$ is compact [2, p. 79] and that $\mathfrak{Q}_1$ is a closed subset of $\mathfrak{R}_1$. The mapping $\sigma: N_1 \rightarrow N_1 \cap A_1$ is a mapping of the compact set $\mathfrak{Q}_1$ into $\mathfrak{R}_1$ which is continuous (see [2, p. 85]). Also $\mathfrak{R}_1$ is a Hausdorff space [2, p. 84] so that $\sigma(\mathfrak{Q}_1)$ is closed in $\mathfrak{R}_1$.

The mapping $\tau: M_1 \rightarrow M_1 \cap A$ is a homeomorphism of $\mathfrak{M}_1 \sim \{A\}$ onto $\mathfrak{M}$. But $\sigma(\mathfrak{Q}_1) \sim \{A\}$ is closed in $\mathfrak{M}_1$ so that $\tau[\sigma(\mathfrak{Q}_1) \sim \{A\}]$ is closed in $\mathfrak{M}$. This set is the same as $E(\mathfrak{Q}) = \{N \cap A|N \in \mathfrak{Q}, N \not\supset A\}$. 

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Recall that, by Theorem 3.1, \( E(\mathcal{D}) \supseteq \Delta(A) \). From our definitions \( \Delta(A) \) is dense in \( \mathcal{Y} \) so that also \( E(\mathcal{Y}) \supseteq \mathcal{Y} \). Since the reverse inequality is clear, the proof is complete.

References


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SEMI-HOMOGENEOUS FUNCTIONS

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1. Introduction and statement of results. A function \( f \) is called **homogeneous of degree \( n \) with respect to the set \( A \)**, or briefly semi-homogeneous if

\[
(1.0) \quad f(ax) = a^nf(x)
\]

is satisfied for all \( x \) in the domain of \( f \) and all \( a \) in \( A \).

With each admissible \( A \) there is associated a class of solutions of (1.0). E.g., let \( R \) denote the set of all real numbers and let \( f \) be a function on \( R \) to \( R \). If \( A \) consists only of the irrationals, then \( f(x) = cx^n \) \((c = f(1))\) is the unique solution of (1.0). On the other hand, if \( A \) consists only of the rationals, then in addition to \( f(x) = cx^n \), (1.0) has other solutions (e.g., if \( n \) is any nonzero integer and \( f(x) = x^n \) or 0 accordingly as \( x \) is rational or irrational).

We are interested in studying decompositions of the set of admissible \( A \)’s into classes and in characterizing the solutions of (1.0) corresponding to these classes. In this paper we show how this can be done in a natural way for semi-homogeneous functions of a real variable. We note that in this case our methods apply to

\[
(1.1) \quad f(ax) = p(a)f(x) \quad (a \in A \subset R),
\]

where \( p \) is a product-preserving function on \( R \) to \( R \) (cf. [1]). We shall therefore confine our attention to (1.1).

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