EXAMPLE OF A NONACYCLIC CONTINUUM SEMIGROUP $S$ WITH ZERO AND $S = ESE$

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Throughout this discussion $S$ will denote a compact connected topological semigroup and $E$ will denote the set of idempotents of $S$. The problem to be considered concerns a question posed by Professor A. D. Wallace. In [1], Wallace proves that if $S$ has a left unit, if $I$ is a closed ideal of $S$, and if $L = \emptyset$ or if $L$ is a closed left ideal of $S$, then $H^n(S) \cong H^n(I \cup L)$ for all integers $n$, where $H^n(A)$ denotes the $n$th Alexander–Čech cohomology group of $A$ with coefficients in an arbitrary but fixed group $G$. If $S$ is assumed to have both a left zero and a left unit, then it follows that each closed left ideal $L$, of $S$ is acyclic; that is, $H^p(L) = 0$ for all $p \geq 1$. A dual statement holds for closed right ideals if $S$ has a right unit and right zero. A generalization of the case in which $S$ has a left, right, or two-sided unit, is to require that $S = ES$, $S = SE$, or $S = ESE$, respectively, and Wallace has asked: “If $S$ has a zero, are closed right or left ideals of $S$ necessarily acyclic in the more general situation?” [3]. A negative answer to this question is given here by way of examples, and a theorem is proved giving a necessary and sufficient condition for closed right ideals of $S$ to be acyclic, assuming $S = ESE$ and $S$ has a zero. Following the proof of this theorem is an example of a semigroup not satisfying this condition.

The above-mentioned example shows that even though $S$ is acyclic, it is not necessarily true that all closed right ideals of $S$ are acyclic. Thus the question remains as to whether $S$ is acyclic if $S = ESE$ and $S$ has a zero [3]. Wallace proves in [2] that for such a semigroup $S$, $H^1(S) = 0$, however, an example is given here of a semigroup $S$ with zero, $S = ESE$ and $H^2(S) \cong G$ for all groups $G$, showing that this question also has a negative answer. Two further examples are included in this paper which show what can occur if one only assumes that $S = SE$, or $S = ES$. One example is of a semigroup $S$ with zero, $S = SE$ and $H^1(S) \cong G$ for all groups $G$ and the other is an example of a semigroup $S$ with zero and left unit and $S$ contains a closed right ideal $R$ with $H^1(R) \cong G$.

DEFINITION. Let $T$ be a semigroup, $a \in T$ and $R(a)$ the closed right ideal of $T$ generated by $a$. Then $a$ is said to be right codependent on

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T if for any integer $n \geq 1$, $H^n(T) = 0$ implies that $H^n(R(a)) = 0$.

**Theorem.** Let $S$ be a compact connected semigroup with zero and $S = ESE$. A necessary and sufficient condition that each closed right ideal of $S$ be acyclic is that each $a$ in $S$ be right codependent on $S$.

The proof of this theorem depends on the following two lemmas. The proofs of these lemmas are paraphrases of the proof of the main theorem in [2] and will be omitted.

**Lemma 1.** Let $S$ be a compact connected semigroup with zero and $S = ES$. Let $n$ be a fixed integer $n \geq 2$. If $H^{n-1}(R) = 0$ for each closed right ideal $R \subseteq S$, then $H^n(S) = 0$.

**Lemma 2.** Let $S$ be a compact connected semigroup with zero and $S = SE$. Let $n$ be a fixed integer, $n \geq 1$. If for each $a \in S$, $H^n(aS) = 0$ and if for each closed subset $A \subseteq S$, $H^{n-1}(A.S) = 0$, then $H^n(R) = 0$ for each closed right ideal $R \subseteq S$. (For $n = 1$, reduced groups are to be used.)

**Proof of Theorem.** First assume that each $a$ in $S$ is right codependent on $S$. The proof of sufficiency will be by induction on $n$. Let $n = 1$. From [2], $H^1(S) = 0$, hence it follows that $H^1(aS) = 0$ for each $a$ in $S$ since $R(a) = aS$ and each $a$ is right codependent on $S$. Each closed right ideal $R \subseteq S$ is connected, therefore $H^0(R, r) = 0$ for each $r \in R$. Thus, using reduced groups, it follows from Lemma 2 that $H^1(R) = 0$ for each closed right ideal $R \subseteq S$.

Assume now that $H^{k-1}(R) = 0$ for each closed right ideal $R$ of $S$ and integer $k \geq 2$. Then by Lemma 1, $H^k(S) = 0$, hence $H^k(aS) = 0$ for each $a \in S$. Applying Lemma 2, it follows that $H^k(R) = 0$ where $R$ is a closed right ideal of $S$. This completes the proof of sufficiency.

If each closed right ideal of $S$ is acyclic, then $H^n(aS) = 0$ for each $a \in S$, integer $n \geq 1$ and coefficient group $G$ since $aS$ is a closed right ideal. Also $R(a) = aS$ for each $a \in S$ so that it is trivially true that each $a$ in $S$ is right codependent on $S$ which completes the proof of the theorem.

In the following examples, let $I = [0, 1]$ denote the real unit interval and for $x$ and $y$ in $I$ let:

- $x \wedge y = \text{minimum of } x \text{ and } y$,
- $x \vee y = \text{maximum of } x \text{ and } y$,
- $xy = \text{real product of } x \text{ and } y$.

**Example 1.** This is an example of a compact connected semigroup $S$ with zero, $S = ESE$, $S$ is acyclic and there exists an element $p$ in $S$ with $H^1(pS) \cong G$ for all groups $G$. This example shows that there exist

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semigroups with zero such that each element is not right codependent on S. The topological space of S is a two-cell with three closed intervals, I1, I2, and I3, issuing from a common point z0, on the boundary, B, of the two-cell. This point z0 is the zero of S and \( p \subset B \setminus z_0 \). By construction \( pS = B \) and \( B^2 = z_0 \). In this example, \( E = \{ e_1, e_2, e_3, z_0 \} \) where \( e_i \) is the free endpoint of \( I_i \).

Example 1 is constructed as follows. Let \( \{ a, b, c, d, \theta \} \) be a discrete space consisting of five elements. Define spaces \( A, B, C, D \) and \( S_0 \) as follows:

\[
A = a \times I, \quad B = b \times I, \quad C = c \times I, \quad D = d \times I \times I, \quad S_0 = A \cup B \cup C \cup D \cup \{ \theta \}
\]

with the topology on \( S_0 \) given by the union of the topologies on \( A, B, C, D \) and \( \{ \theta \} \). Define the product \( pq \) for \( p \) and \( q \) in \( S_0 \) by:

\[
\begin{align*}
pq & = (d, (x \lor y)r, (x \lor y)), & \text{if } p = (d, x, y) \in D, q = (a, r) \in A, \\
& = (d, (x \lor y), (x \lor y)r), & \text{if } p = (d, x, y) \in D, q = (b, r) \in B, \\
& = (b, rs), & \text{if } p = (a, r) \in A, \text{ or } p = (b, r) \in B \text{ and } q = (b, s) \in B, \\
& = (a, rs), & \text{if } p = (a, r) \in A, \text{ or } p = (b, r) \in B \text{ and } q = (a, s) \in A, \\
& = (d, x, yr), & \text{if } p = (c, r) \in C \text{ and } q = (d, x, y) \in D, \\
& = (c, rs), & \text{if } p = (c, r) \in C \text{ and } q = (c, s) \in C, \\
& = \theta, & \text{otherwise.}
\end{align*}
\]

By the definition of the topology on \( S_0 \), multiplication is continuous and associativity is checked by direct computation. Let \( E_0 = \{ (a, 1), (b, 1), (c, 1), \theta \} \). Then \( E_0 \) is a set of idempotents in \( S_0 \) and the claim is made that \( S_0 = E_0S_0E_0 \). This is true since \( (a, 1) \) is a two-sided unit for \( A \) and a right unit for \( (d \times \{ x \mid x \leq y \}) \); \( (b, 1) \) is a two-sided unit for \( B \) and a right unit for \( (d \times \{ x \mid x \geq y \}) \); \( (c, 1) \) is a two-sided unit for \( C \) and a left unit for \( D \); and \( \theta^2 = \theta \).

Consider now, \( I_0 = (d \times \{ 0 \} \times I) \cup (d \times I \times \{ 0 \}) \cup \{ (a, 0), (b, 0), (c, 0), \theta \} \). By direct computation it can be shown that this closed subset of \( S_0 \) is a two-sided ideal of \( S \). Let \( S = S_0/I_0 \) be the Rees quotient of \( S_0 \) by \( I_0 \). Then \( S \) is a compact connected semigroup with zero and it is clear that \( S \) is acyclic. Also the condition \( S_0 = E_0S_0E_0 \) implies that \( S = ESE \) where \( E \) is the set of idempotents of \( S \).

Let \( p = (d, 1, 1) \). Then \( pS = ((d, 1, 1)S_0 \cup I_0)/I_0 = ((d \times I \times \{ 1 \}) \cup (d \times \{ 1 \} \times I) \cup I_0)/I_0 \) so that \( pS \) is homeomorphic to a one-sphere and therefore \( H^1(pS) \cong G \) for all groups \( G \).

Example 2. This is an example of a compact connected semigroup \( S \) with zero, \( S = ESE \) and \( H^2(S) \cong G \) for all coefficient groups \( G \). The topological space of this semigroup is a two-sphere with four closed intervals, \( I_i, i = 1, 2, 3, 4 \), issuing from a common point, \( z_0 \), on the two-
sphere. The point \( z_1 \) is a zero for \( S \) and if \( e_i \) denotes the free endpoint of \( I_i \), then \( E = \{ e_1, e_2, e_3, z_1 \} \) and multiplication in \( S \) has the following properties: (Let \( S_1 \) denote the two-sphere in \( S \); \( C_1 \) and \( C_2 \) the two great circles in \( S_1 \) through \( z_1 \); \( H_1 \) and \( H_2 \) the closed hemispheres determined by \( C_1 \); and \( P_1, P_2 \) the closed hemispheres determined by \( C_2 \). ) \( S_1^2 = z_1 \); \( e_1 S = H_1 \cup I_1 \); \( e_2 S = H_2 \cup I_2 \); \( S e_3 = P_1 \cup I_3 \); \( S e_4 = P_2 \cup I_4 \). Hence \( e_3 S \cap e_4 S = C_1 \) is a closed right ideal of \( S \) with nontrivial cohomology in dimension one. Similarly, \( S e_3 \cap S e_4 = C_2 \) is a closed left ideal of \( S \).

\( S \) is constructed in the following way: Let \( N = \{ a, b, c, d, e, \theta \} \) be a discrete space with six elements. Let \( N_0 = N \setminus \{ e, \theta \} \) and let \( T = (N_0 \times I) \cup (e \times I \times I) \cup \{ \theta \} \) with the topology on \( T \) given by the union of the topologies of its subsets. For \( x \) and \( y \) in \([0, 1]\) define \( \alpha(x, y) = (x \wedge y \wedge (1-x) \wedge (1-y)) \) and define the product \( pq \) for \( p \) and \( q \) in \( T \) by:

\[
\begin{align*}
(p, (e, (x \wedge (1-y)) - r\alpha(x, y)), (y \wedge (1-x)) - r\alpha(x, y)), & \quad p = (a, r) \in A, \\
& \quad q = (e, (x, y)) \in S_0, = e \times I \times I \\
(e, (x \vee (1-y)) + r\alpha(x, y), (y \vee (1-x)) + r\alpha(x, y)), & \quad p = (b, r) \in B, \\
& \quad q = (e, (x, y)) \in S_0 \\
(a, r+s-rs), & \quad p = (a, r) \in A \text{ and } q = (a, s) \in A \text{ or } q = (b, s) \in B, \\
(b, r+s-rs), & \quad p = (b, r) \in B \text{ and } q = (a, s) \in A \text{ or } q = (b, s) \in B, \\
(e, 1-s+(x \vee y)s, (x \wedge y)s), & \quad p = (e, x, y) \in S_0, q = (c, s) \in C, \\
(e, (x \wedge y)s, 1-s+(x \vee y)s), & \quad p = (e, x, y) \in S_0, q = (d, s) \in D, \\
(d, rs), & \quad p = (c, r) \in C \text{ or } p = (d, r) \in D \text{ and } q = (d, s) \in D, \\
(c, rs), & \quad p = (d, r) \in D \text{ or } p = (c, r) \in C \text{ and } q = (c, s) \in C, \\
& \quad \theta \text{ otherwise.}
\end{align*}
\]

By the definition of the topology on \( T \), it is clear that multiplication is continuous since it involves continuous operations of real numbers. By direct computation it is seen that multiplication is also associative and therefore \( T \) is a compact semigroup. Let \( S_i, i = 1, 2, 3, 4 \), subsets of \( e \times I \times I \) be defined by:

\[
\begin{align*}
S_1 & = \{(e, x, y) : 0 \leq x \leq y \leq x + y \leq 1 \}, \\
S_2 & = \{(e, x, y) : 0 \leq x \leq y \leq 1 \leq x + y \}, \\
S_3 & = \{(e, x, y) : 0 \leq y \leq x \leq 1 \leq x + y \}, \\
S_4 & = \{(e, x, y) : 0 \leq y \leq x \leq x + y \leq 1 \},
\end{align*}
\]

and let \( E_0 = \{(a, 0), (b, 0), (c, 1), (d, 1), \theta \} \). \( E_0 \) is a set of idempotents.
in $T$ and the claim is made that $T = E_0 E_0$. This follows from the following equalities:

$$a \times I = (a, 0)(a \times I)(a, 0); \quad b \times I = (b, 0)(b \times I)(b, 0); \quad c \times I = (c, 1)(c \times I)(c, 1); \quad d \times I = (d, 1)(d \times I)(d, 1); \quad \theta^2 = \theta \quad \text{and} \quad e \times I \times I = \bigcup \{S_i: i = 1, 2, 3, 4\} = (a, 0)S_i(d, 1) \cup (b, 0)S_i(d, 1) \cup (b, 0)S_i(c, 1) \cup (a, 0)S_i(c, 1) \}. \] This proves that $T = E_0 E_0$ as claimed.

Now let $I_0 = \{(a, 1), (b, 0), (c, 0), (d, 0), \theta\} \cup (e \times F(I \times I))$ where $F(I \times I)$ denotes the boundary of $I \times I$ in the Euclidean plane. It can be shown that this closed subset of $T$ is a two-sided ideal of $T$, hence $S = T/I_0$ is a compact connected semigroup as described above. Also $S = ESE$, since $T = E_0 E_0$ and $S$ has a zero.

**Example 3.** This example is of a semigroup $S = SE$ which is compact connected, has a zero and $H^1(S) \cong G$ for all groups $G$. $S$ is a subsemigroup of the semigroup in Example 1 and the topological space of $S$ is a circle with two closed intervals issuing from a common point of the circle.

In the terminology of Example 1, consider the following closed subsemigroup, $T$, of $S_0$:

$$T = A \cup B \cup (d \times F(I \times I)) \cup \{\theta\}. \] Then $T = T(a, 1) \cup T(b, 1) \cup \{\theta\}$ so that $T = T E_1$ where $E_1$ is the set of idempotents in $T$. Let $I_1 = \{(a, 0), (b, 0), \theta\} \cup (d \times \{0\} \times I) \cup (d \times I \times \{0\}) \cup (e \times F(I \times I))$.

**Example 4.** This final example is of a semigroup $S$ with zero and left unit and $S$ contains a closed right ideal $R$ such that $H^1(R) \cong G$, for all groups $G$.

Let $S = \{0\} \times I \times I \cup (I \times \{0\} \times I)$ and define multiplication in $S$ by $(x, y, z)(r, s, t) = (xr, xs, zt)$. $S$ can be represented by the following matrix semigroup:

$$\begin{pmatrix} x & y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} : (x, y, z) \in S$$

so that multiplication in $S$ is continuous and associative. Clearly $(0, 0, 0)$ is a zero for $S$ and $(1, 0, 1)$ is a left unit for $S$. By the definition of multiplication it follows that any subset of $\{0\} \times I \times I$ containing $(\{0\} \times \{0\} \times I)$ is a right ideal of $S$ and, in particular, $R$ = the boundary of $(\{0\} \times I \times I)$ is a closed right ideal of $S$ and $H^1(R) \cong G$ for all coefficient groups $G$.

In these four examples it might be noted that the set of idem-
potents in each semigroup was a finite discrete set. It might be of interest to know if there exists a semigroup \( S = ESE \) which is compact connected, has a zero, is not acyclic and such that the set of idempotents is connected.

**Bibliography**


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**CONFORMAL VECTOR FIELDS IN COMPACT RIEMANNIAN MANIFOLDS**

T. K. Pan

1. **Introduction.** Let \( V^n \) be a compact Riemannian manifold of dimension \( n \) and of class \( C^2 \). Let \( g_{ij}(x) \) of class \( C^2 \) be the coefficients of the fundamental metric which is assumed to be positive definite. Let \( \Gamma^t_{ij} \) be the Christoffel symbol, \( R_{ijkh} \) the curvature tensor and \( R_{ij} \) the Ricci tensor.

Let \( \phi \) be an arbitrary scalar invariant, \( \xi^i \) an arbitrary vector field and \( \xi_{i_1i_2\ldots i_p} \) an arbitrary anti-symmetric tensor field of order \( p \), all of class \( C^2 \) in \( V^n \). We shall make use of the following results obtained by S. Bochner and K. Yano [1, pp. 31, 51, 69]:

(1.1) \( (\Delta \phi \geq 0 \) everywhere in \( V^n \)) \( \Rightarrow \) (\( \phi = \) constant everywhere in \( V^n \)).

(1.2) \[
\int_{V^n} \xi^i_{;i} \, dv = 0.
\]

(1.3) \[
\int_{V^n} (R_{ij} \xi^i_{;j} + \xi^i_{;j} \xi^i_{;i} - \xi^i_{;i} \xi^i_{;j}) \, dv = 0.
\]

(1.4) \[
\int_{V^n} (\xi_{i_1i_2\ldots i_p} + \xi_{i_2\ldots i_p} + \xi_{i_1i_2\ldots i_p} - \xi_{i_2\ldots i_p} \xi_{i_1} \xi_{i_2} \ldots i_p) \, dv = 0
\]

where