$$T_n(z) \sim n!\left(\frac{e^z}{nz}\right)^n \phi_J(z) - \sum_{r=0}^{\infty}\left(\frac{1}{n}\right)^r U_r(z). \tag{6}$$

If $z=-1$, (5) reduces to Copson's series. Ramanujan's approximation for $S_n(1)$ is, in view of (5), a considerably more singular result than it would otherwise appear.

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ON THE LOCAL LINEARIZATION OF DIFFERENTIAL EQUATIONS[1]

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1. Consider the autonomous system of real, nonlinear differential equations

$$x' = Ex + F(x), \quad \text{where} \quad F(|x|) = o(|x|) \text{ as } x \to 0, \tag{1.1}$$

x is a (Euclidean) vector, $F(x)$ a smooth vector-valued function of x, and E a constant matrix with eigenvalues $e_1, e_2, \cdots$ satisfying

$$\operatorname{Re} e_j \neq 0. \tag{1.2}$$

Let the solution $\xi(t, x)$ of (1.1) starting at x for $t=0$ be written as

$$T^t: x^t = \xi(t, x) = e^{Et}x + X(t, x), \tag{1.3}$$

where $X(t, x)=o(|x|)$ as $x\to 0$ (for fixed t). Thus if T^t is considered as a map $x\to x^t$, for fixed t, the composition rule

$$T^tT^s = T^{t+s} \tag{1.4}$$

is valid for small $|x|$. Correspondingly, the linear system

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$$u' = Eu \tag{1.5}$$

leads to the linear maps

$$L^t\colon u^t = e^{Et}u. \tag{1.6}$$

The object of this note is to prove

(*) *In* (1.1), *let* $F(x)$ *be of class* C^1 *for small* $|x|$ (*or uniformly Lipschitz continuous with a Lipschitz constant for* $|x| \leqq \epsilon$ *which tends to* 0 *as* $\epsilon \to 0$) *and let the eigenvalues of* E *satisfy* (1.2). *Let* $x(t) = \xi(t, x)$ *in* (1.3) *be the solution of* (1.1) *satisfying* $x(0) = x$. *Then there exists a topological map*

$$R\colon u = u(x) \tag{1.7}$$

of a neighborhood of $x = 0$ *onto a neighborhood of* $u = 0$ *such that*

$$RT^tR^{-1} = L^t. \tag{1.8}$$

The analogue of this result was proved in [2] under the assumption that F is of class C^2. The proof here will be simpler than that of [2] by avoiding the use of Theorem (II) of [1] and by following the procedure of Sternberg [3, p. 817] for passing from the linearization of T^1 to that of T^t. This is accomplished by carrying out some of the normalizations, used in [2] for T^1, for the entire "group" T^t and by an examination of the stable and unstable manifolds of T^t.

The proof will be given only for the case that F is of class C^1; the arguments for the case that F is uniformly Lipschitz continuous are analogous.

2. It will be supposed that not all e_j satisfy $\mathrm{Re}\, e_j > 0$ or $\mathrm{Re}\, e_j < 0$, as this case is simple (cf. [2]). After a suitable linear change of variables, it can be supposed that E is in a suitable normal form (to be specified by (2.3) and (2.4), below).

Since the assumption and assertion of (*) are local, nothing is changed if $F(x)$ is altered outside of a small vicinity of $x = 0$. Thus there is no loss of generality in supposing that if $\theta > 0$ is arbitrary, then there is an $s = s_\theta > 0$ such that $s_\theta \to 0$ as $\theta \to 0$, $F(x)$ is defined for all x, $F(x) = 0$ for $|x| \geqq s$, and that the norm $|F_x|$ of the Jacobian matrix F_x does not exceed θ for all x. For otherwise, $F(x)$ can be replaced by $F(x)\phi(|x|)$, where $\phi(t)$ is a smooth function for $t \geqq 0$ satisfying $\phi(t) = 1$ for $0 \leqq t \leqq s/2$, $\phi(t) = 0$ for $t \geqq s$, $|d\phi/dt| \leqq 3/s$ and $s > 0$ is sufficiently small (cf. [2, p. 612]). Thus T^tx is defined for all x and $-\infty < t < \infty$ and is indeed a group of maps.

It follows from (1.1) and $F(x) = 0$ for $|x| \geqq s$ that there exists an

$r=r_\theta>0$ such that $r_\theta\to 0$ as $\theta\to 0$ and the general solution (1.3) of (1.1) satisfies

$$x^t = e^{Et}x \qquad \text{if } |x| \geqq r,\ 0 \leqq t \leqq 1. \tag{2.1}$$

In fact, $r>0$ can be any number such that a solution of (1.1) beginning for $t=0$ at x, $|x|\geqq r$, does not enter $|x|\leqq s$ for $0\leqq t\leqq 1$.

There also exists an $\epsilon=\epsilon_\theta$ such that $\epsilon_\theta\to 0$ as $\theta\to 0$ and the Jacobian matrix X_x of $X(t, x)$ in (1.3) satisfies

$$|X_x(t, x)| \leqq \epsilon \qquad \text{for all } x \text{ and } 0 \leqq t \leqq 1. \tag{2.2}$$

In view of (1.3) and (2.1), it is sufficient to verify (2.2) only for $|x|\leqq r$, where $x^t=\xi(t, x)$ is uniformly bounded for $0\leqq t\leqq 1$. The Jacobian matrix $\partial x^t/\partial x=e^{Et}+X_x$ satisfies a linear differential equation and the variation of constants $\partial x^t/\partial x=e^{Et}J$ defines a matrix $J=J(t, x)$ satisfying the differential equation $J'=e^{-Et}F_x(x^t)e^{Et}J$ and the initial condition $J=I$ at $t=0$, where I is the unit matrix. It is clear that J is uniformly bounded for $|x|\leqq r$, $0\leqq t\leqq 1$ (and $0<\theta\leqq 1$). Since $|F_x|\leqq\theta$, there is a constant c such that $|J'|\leqq c\theta$ for $0\leqq t\leqq 1$ and so $|J(t, x)-I|\leqq c\theta$ for $0\leqq t\leqq 1$. Hence $\epsilon=\epsilon_\theta$ can be chosen to be $c\theta \sup |e^{Et}|$, $0\leqq t\leqq 1$.

By assuming that E is in a suitable normal form and writing (x, y) in place of x, $(X(t, x, y), Y(t, x, y))$ in place of $X(t, x)$, etc., (1.3) can be written in the form

$$T^t\colon x^t = e^{\Gamma t}x + X(t, x, y), \qquad y^t = e^{\Delta t}y + Y(t, x, y), \tag{2.3}$$

where x, y are vectors; Γ, Δ matrices with eigenvalues satisfying $\operatorname{Re}\gamma_j<0$, $\operatorname{Re}\delta_k>0$, respectively; and X, Y are of class C^1 for all x, y and t. It can also be supposed that there are numbers a, b such that

$$|e^\Gamma| \leqq a < 1 \quad \text{and} \quad |e^{-\Delta}| \leqq 1/b < 1. \tag{2.4}$$

It is clear from (1.3) that

$$X, Y = o(|x| + |y|) \quad \text{as} \quad x, y \to 0 \ (\text{for fixed } t) \tag{2.5}$$

and from (2.1), (2.2) that

$$X, Y = 0 \qquad \text{for } |x|^2 + |y|^2 \geqq r^2 \text{ and } 0 \leqq t \leqq 1, \tag{2.6}$$

$$|X_x|,\ |X_y|,\ |Y_x|,\ |Y_y| \leqq \epsilon \quad \text{for } 0 \leqq t \leqq 1 \text{ and all } x, y. \tag{2.7}$$

Corresponding to the form (2.3) of (1.3), the linear map (1.6) has the form

$$L^t\colon u^t = e^{\Gamma t}u, \qquad v^t = e^{\Delta t}v. \tag{2.8}$$

3. Let $M_\pm$ be the set of points $\{(x, y) : T^t(x, y)\to(0, 0) \text{ as } t\to\pm\infty\}$.

Then, as is known, the subsets of $M_\pm$ in a sufficiently small vicinity of $x=y=0$, are manifolds of class C^1 of the respective forms $y=\eta_0(x)$, $x=\xi_0(y)$. The proof of this (local) assertion in [1, §7] can be adapted to show that if r, ϵ are sufficiently small in (2.6)–(2.7), then, under the normalizations (2.3)–(2.7), the sets $M_\pm$ are C^1 manifolds of the forms $y=\eta_0(x)$, $x=\xi_0(y)$, defined for all x, y, respectively, with $\xi_0(0)=0$, $\xi_{0y}(0)=0$, $\eta_0(0)=0$, $\eta_{0y}(0)=0$; also ξ_0, η_0 and their Jacobian matrices have a bound, say δ, for all x, y, such that $\delta\to 0$ as ϵ, $r\to 0$.

It is clear that if δ is small enough, then

$$M_+ \cap M_- = (0, 0). \tag{3.1}$$

It will also be verified that

$$(x, y) \notin M_+ \Rightarrow |x^n| \to 0, \ |y^n| \to \infty \quad \text{as} \quad n \to \infty, \tag{3.2}$$

$$(x, y) \notin M_- \Rightarrow |x^n| \to \infty, \ |y^n| \to 0 \quad \text{as} \quad n \to -\infty. \tag{3.3}$$

In order to obtain (3.2), consider the maps

$$R_0\colon u = x,\ v = y - \eta_0(x); \qquad R_0^{-1}\colon x = u,\ y = v + \eta_0(u) \tag{3.4}$$

and

$$R_0T^1R_0^{-1}\colon u^1 = e^{\Gamma}u + U_0(u, v), \qquad v^1 = e^{\Delta}v + V_0(u, v), \tag{3.5}$$

where U_0, V_0 and their Jacobians are small for all (u, v). In particular let $|V_{0v}| \leqq \delta_0$, where $\delta_0\to 0$ as ϵ, $r\to 0$. In (u, v)-coordinates, $M_+\colon v=0$ is an invariant manifold so that $V_0(u, 0)=0$. Thus $|V_0(u, v)| \leqq \delta_0|v|$ and $|v^1| \geqq |e^{\Delta}v| - \delta_0|v| \geqq (b-\delta_0)|v|$, by (2.4). Hence $|v^n| \geqq (b-\delta_0)^n|v|$, where $b-\delta_0 > 1$ if δ_0 is sufficiently small. If $(x, y) \notin M_+$ and (u, v) is given by (3.4), then $v\neq 0$ and $|v^n|\to\infty$ as $n\to\infty$. The R_0^{-1} image of (u^n, v^n) is (x^n, y^n) and $y^n = v^n + \eta_0(u^n)$. The boundedness of $\eta_0(u)$ gives $|y^n|\to\infty$ as $n\to\infty$. For large n, (2.6) and (2.3) show that $|x^n| = |e^{\Gamma}x^{n-1}| \leqq a|x^{n-1}|$. Hence $|x^n|\to 0$ as $n\to\infty$. This gives the relations (3.2). The relations (3.3) are verified in the same way.

4. Consider the problem of linearizing T^1, that is, the problem of the existence of a continuous, one-to-one map of the (x, y)-space onto the (u, v)-space

$$R_1\colon u = U_1(x, y), \qquad v = V_1(x, y) \tag{4.1}$$

such that

$$L^1R_1 = R_1T^1. \tag{4.2}$$

The existence of such an R_1 was proved in [2] under the additional normalizations $X(1, 0, y)=0$, $Y(1, x, 0)=0$ (i.e., $M_+\colon y=0$ and

$M_-: x=0$). But these normalizations are inessential.

The relation (4.2) is equivalent to a functional equation

$$(4.3) \qquad e^{\Delta}V_1(x, y) = V_1(e^{\Gamma}x + X(1, x, y), e^{\Delta}y + Y(1, x, y))$$

for V_1 and a similar one for U_1. In order to obtain a solution for V_1, consider the successive approximations

$$(4.4_0) \quad V^0(x, y) = y,$$

$$(4.4_n) \quad V_n(x, y) = e^{-\Delta}V^{n-1}(e^{\Gamma}x + X(1, x, y), e^{\Delta}y + Y(1, x, y)) \text{ for } n \geqq 1.$$

If r, ϵ in (2.6), (2.7) are sufficiently small, a simple induction shows the existence of K, δ, η, with $0<\delta$, $\eta<1$, such that $|V^n - V^{n-1}| \leqq K\eta^n(|x|+|y|)^\delta$ for $n=1, 2, \cdots$; [2, p. 613]. Hence $V_1(x, y) = \lim V^n(x, y)$, $n\to\infty$, exists uniformly on bounded (x, y)-sets. The existence of a continuous $U_1(x, y)$ is obtained similarly.

It follows that there is a continuous map (4.1) satisfying (4.2). It therefore only remains to show that R_1 is one-to-one and onto the (u, v)-space. Note that from (2.4), (2.6), (4.4_n) and corresponding formulae for the successive approximations of U_1, it is seen that

$$(4.5) \quad R_1(0, 0) = (0, 0),$$

$$(4.6) \quad U_1(x, y) = x \text{ if } |x| \geqq r/a \text{ and } V_1(x, y) = y \text{ if } |y| \geqq r.$$

Suppose, if possible, that there exists a pair of points (x_1, y_1), (x_2, y_2) satisfying

$$(4.7) \qquad R_1(x_1, y_1) = R_1(x_2, y_2) \quad \text{but} \quad (x_1, y_1) \neq (x_2, y_2).$$

It follows from (4.2) and a simple induction that

$$(4.8) \qquad R_1(\overset{n}{x_1}, \overset{n}{y_1}) = R_1(\overset{n}{x_2}, \overset{n}{y_2}) \qquad \text{for } n = 0, \pm 1, \cdots,$$

where $(x_j^n, y_j^n) = T^n(x_j, y_j)$. Since T^t is one-to-one by (2.3) and (2.7),

$$(4.9) \qquad (\overset{n}{x_1}, \overset{n}{y_1}) \neq (\overset{n}{x_2}, \overset{n}{y_2}) \qquad \text{for } n = 0, \pm 1, \cdots.$$

Consider first the case (i) that either (x_1, y_1) or (x_2, y_2) is on an unstable or stable manifold $M_\pm$, say, $(x_1, y_1)\in M_+$. Then $(x_2, y_2)\in M_+$. For otherwise, $(x_1^n, y_1^n)\to(0, 0)$ and, by (3.2), $|y_2^n|\to\infty$, as $n\to\infty$, so that $R_1(x_1^n, y_1^n)\to 0$, by (4.5), and $|R_1(x_2^n, y_2^n)|\to\infty$, by (4.6), as $n\to\infty$. This contradicts (4.8). Now $(x_j, y_j)\in M_+$ for $j=1, 2$ implies that $y_j^n = \eta_0(x_j^n)$ for $j=1, 2$, where $M_+: y=\eta_0(x)$. It follows from (3.1), (3.3) that $|x_j^n|\to\infty$ as $n\to-\infty$. But then (4.6) and (4.8) imply that $x_1^n = x_2^n$ if $-n$ is large. Hence $y_1^n=\eta_0(x_1^n)$ is the same as $y_2^n=\eta_0(x_2^n)$ for large $-n$. This contradicts (4.9)and shows that case (i) cannot occur.

Consider the case (ii) that neither (x_1, y_1) nor (x_2, y_2) is on $M_\pm$. Then $|x_j^n| \to 0$, $|y_j^n| \to \infty$ as $n \to \infty$ and $|x_j^n| \to \infty$, $|y_j^n| \to 0$ as $n \to -\infty$ for $j=1, 2$ (cf. (3.2)–(3.3)). The argument in [2, p. 614] shows that case (ii) is impossible if a, b, ϵ in (2.4), (2.7) satisfy $a+2\epsilon<1<b-2\epsilon$. Hence R_1 is one-to-one. It is an open map by invariance of domain. It is also easy to see from (4.6) that the range of R is closed. Hence R is onto.

5. It was observed by Sternberg [3, p. 817] that (4.2) implies that the map

$$R = \int_0^1 L^{-t} R_1 T^t dt \tag{5.1}$$

satisfies, for all t,

$$L^t R = RT^t. \tag{5.2}$$

Thus, in order to complete the proof of (*), i.e., of (1.8), it only remains to verify that the map (5.1) is one-to-one and onto.

In view of the normalizations above for T^t, $0 \leqq t \leqq 1$, this verification turns out to be the same as the verification that R_1 is one-to-one. In fact, if (5.1) is written as

$$R\colon u = U(x, y), \qquad v = V(x, y), \tag{5.3}$$

then

$$R(0, 0) = (0, 0). \tag{5.4}$$

Also, by (5.1) and (2.3), (2.6), (4.8), there is a constant $c>0$ such that

$$U(x, y) = x \text{ if } |x| \geqq c, \qquad V(x, y) = y \text{ if } |y| \geqq c. \tag{5.5}$$

Thus, analogues of the relations (4.2), (4.5), (4.6) which were used to show that R_1 is one-to-one hold for R, namely, (5.2), (5.4), (5.5). It follows that R is one-to-one and onto. This completes the proof of (*).

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