If \( z = -1 \), (5) reduces to Copson's series. Ramanujan's approximation for \( S_\infty(1) \) is, in view of (5), a considerably more singular result than it would otherwise appear.

**References**


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**ON THE LOCAL LINEARIZATION OF DIFFERENTIAL EQUATIONS**

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1. Consider the autonomous system of real, nonlinear differential equations

\[
(1.1) \quad x' = Ex + F(x), \quad \text{where} \quad F(|x|) = o(|x|) \text{ as } x \to 0,
\]

\( x \) is a (Euclidean) vector, \( F(x) \) a smooth vector-valued function of \( x \), and \( E \) a constant matrix with eigenvalues \( e_1, e_2, \ldots \) satisfying

\[
(1.2) \quad \Re e_i \neq 0.
\]

Let the solution \( \xi(t, x) \) of (1.1) starting at \( x \) for \( t = 0 \) be written as

\[
(1.3) \quad T^t : x^t = \xi(t, x) = e^{Et}x + X(t, x),
\]

where \( X(t, x) = o(|x|) \) as \( x \to 0 \) (for fixed \( t \)). Thus if \( T^t \) is considered as a map \( x \to x^t \), for fixed \( t \), the composition rule

\[
(1.4) \quad T'T^s = T^{t+s}
\]

is valid for small \( |x| \). Correspondingly, the linear system

\[
(6) \quad T_n(z) \sim n! \left( \frac{e^z}{n!} \right)^n \phi_n(z) - \sum_{r=0}^{\infty} \left( \frac{1}{n} \right)^r U_r(z).
\]
leads to the linear maps

\[ L^t: u^t = e^{\lambda t} u \]

The object of this note is to prove

(*) In (1.1), let \( F(x) \) be of class \( C^1 \) for small \( |x| \) (or uniformly Lipschitz continuous with a Lipschitz constant for \( |x| \leq \varepsilon \) which tends to 0 as \( \varepsilon \to 0 \)) and let the eigenvalues of \( E \) satisfy (1.2). Let \( x(t) = \xi(t, x) \) in (1.3) be the solution of (1.1) satisfying \( x(0) = x \). Then there exists a topological map

\[ R: u = u(x) \]

of a neighborhood of \( x = 0 \) onto a neighborhood of \( u = 0 \) such that

\[ RT^t R^{-1} = L^t. \]

The analogue of this result was proved in [2] under the assumption that \( F \) is of class \( C^2 \). The proof here will be simpler than that of [2] by avoiding the use of Theorem (II) of [1] and by following the procedure of Sternberg [3, p. 817] for passing from the linearization of \( T^t \) to that of \( T^t \). This is accomplished by carrying out some of the normalizations, used in [2] for \( T^t \), for the entire "group" \( T^t \) and by an examination of the stable and unstable manifolds of \( T^t \).

The proof will be given only for the case that \( F \) is of class \( C^1 \); the arguments for the case that \( F \) is uniformly Lipschitz continuous are analogous.

2. It will be supposed that not all \( e_i \) satisfy \( \text{Re } e_i > 0 \) or \( \text{Re } e_i < 0 \), as this case is simple (cf. [2]). After a suitable linear change of variables, it can be supposed that \( E \) is in a suitable normal form (to be specified by (2.3) and (2.4), below).

Since the assumption and assertion of (*) are local, nothing is changed if \( F(x) \) is altered outside of a small vicinity of \( x = 0 \). Thus there is no loss of generality in supposing that if \( \theta > 0 \) is arbitrary, then there is an \( s = s_0 > 0 \) such that \( s_0 \to 0 \) as \( \theta \to 0 \), \( F(x) \) is defined for all \( x \), \( F(x) = 0 \) for \( |x| \geq s \), and that the norm \( |F_x| \) of the Jacobian matrix \( F_x \) does not exceed \( \theta \) for all \( x \). For otherwise, \( F(x) \) can be replaced by \( F(x)\phi(|x|) \), where \( \phi(t) \) is a smooth function for \( t \geq 0 \) satisfying \( \phi(t) = 1 \) for \( 0 \leq t \leq s/2 \), \( \phi(t) = 0 \) for \( t \geq s \), \( |d\phi/dt| \leq 3/s \) and \( s > 0 \) is sufficiently small (cf. [2, p. 612]). Thus \( T^t x \) is defined for all \( x \) and \( -\infty < t < \infty \) and is indeed a group of maps.

It follows from (1.1) and \( F(x) = 0 \) for \( |x| \geq s \) that there exists an
$r = r_0 > 0$ such that $r_0 \to 0$ as $\theta \to 0$ and the general solution (1.3) of (1.1) satisfies

\[(2.1) \quad x' = e^{rt}x \quad \text{if} \quad |x| \geq r, \quad 0 \leq t \leq 1.\]

In fact, $r > 0$ can be any number such that a solution of (1.1) beginning for $t=0$ at $x$, $|x| \geq r$, does not enter $|x| \leq s$ for $0 \leq t \leq 1$.

There also exists an $\epsilon = \epsilon_0$ such that $\epsilon_0 \to 0$ as $\theta \to 0$ and the Jacobian matrix $X_x$ of $X(t, x)$ in (1.3) satisfies

\[(2.2) \quad |X_x(t, x)| \leq \epsilon \quad \text{for all} \quad x \quad \text{and} \quad 0 \leq t \leq 1.\]

In view of (1.3) and (2.1), it is sufficient to verify (2.2) only for $|x| \leq r$, where $x' = \xi(t, x)$ is uniformly bounded for $0 \leq t \leq 1$. The Jacobian matrix $\partial x'/\partial x = e^{rt} + X_x$ satisfies a linear differential equation and the variation of constants $\partial x'/\partial x = e^{rt}J$ defines a matrix $J = J(t, x)$ satisfying the differential equation $J' = e^{rt}F_x(x')e^{rt}J$ and the initial condition $J = I$ at $t = 0$, where $I$ is the unit matrix. It is clear that $J$ is uniformly bounded for $|x| \leq r$, $0 \leq t \leq 1$ (and $0 < \theta \leq 1$).

Since $|F_x| \leq \theta$, there is a constant $c$ such that $|J' | \leq c \theta$ for $0 \leq t \leq 1$ and so $|J(t, x) - I| \leq c \theta$ for $0 \leq t \leq 1$. Hence $\epsilon = \epsilon_0$ can be chosen to be $c \theta$.

By assuming that $E$ is in a suitable normal form and writing $(x, y)$ in place of $x$, $(X(t, x, y), Y(t, x, y))$ in place of $X(t, x)$, etc., (1.3) can be written in the form

\[(2.3) \quad T': \quad x' = e^{rt}x + X(t, x, y), \quad y' = e^{rt}y + Y(t, x, y),\]

where $x, y$ are vectors; $\Gamma, \Delta$ matrices with eigenvalues satisfying $\text{Re} \gamma_0 < 0, \text{Re} \delta_k > 0$, respectively; and $X, Y$ are of class $C^1$ for all $x, y$ and $t$. It can also be supposed that there are numbers $a, b$ such that

\[(2.4) \quad |e^t| \leq a < 1 \quad \text{and} \quad |e^{-A}| \leq 1/b < 1.\]

It is clear from (1.3) that

\[(2.5) \quad X, Y = o(|x| + |y|) \quad \text{as} \quad x, y \to 0 \quad \text{(for fixed} \quad t)\]

and from (2.1), (2.2) that

\[(2.6) \quad X, Y = 0 \quad \text{for} \quad |x|^2 + |y|^2 \geq r^2 \quad \text{and} \quad 0 \leq t \leq 1,\]

\[(2.7) \quad |X_x|, |X_y|, |Y_x|, |Y_y| \leq \epsilon \quad \text{for} \quad 0 \leq t \leq 1 \quad \text{and} \quad \text{all} \quad x, y.\]

Corresponding to the form (2.3) of (1.3), the linear map (1.6) has the form

\[(2.8) \quad L': \quad u' = e^{rt}u, \quad v' = e^{rt}v.\]

3. Let $M_{\pm}$ be the set of points \{(x, y): T^t(x, y) \to (0, 0) \text{ as} \quad t \to \pm \infty \}.\]
Then, as is known, the subsets of $M_{\pm}$ in a sufficiently small vicinity of $x=y=0$, are manifolds of class $C^1$ of the respective forms $y=\eta_0(x)$, $x=\xi_0(y)$. The proof of this (local) assertion in [1, §7] can be adapted to show that if $r, \epsilon$ are sufficiently small in (2.6)--(2.7), then, under the normalizations (2.3)--(2.7), the sets $M_{\pm}$ are $C^1$ manifolds of the forms $y=\eta_0(x)$, $x=\xi_0(y)$, defined for all $x, y$, respectively, with $\xi_0(0)=0$, $\eta_0(0)=0$, $\eta_0(0)=0$; also $\xi_0, \eta_0$ and their Jacobian matrices have a bound, say $\delta$, for all $x, y$, such that $\delta \to 0$ as $\epsilon, r \to 0$.

It is clear that if $\delta$ is small enough, then

$$M_{+} \cap M_{-} = (0, 0).$$

It will also be verified that

$$(x, y) \in M_{+} \Rightarrow |x^n| \to 0, |y^n| \to \infty \text{ as } n \to \infty,$$

$$(x, y) \in M_{-} \Rightarrow |x^n| \to \infty, |y^n| \to 0 \text{ as } n \to -\infty.$$

In order to obtain (3.2), consider the maps

$$(3.4)\quad R_0: \ u = x, \ v = y - \eta_0(x); \quad R_0^{-1}: \ x = u, \ y = v + \eta_0(u),$$

and

$$(3.5)\quad R_0T^1R_0^{-1}: \ u^1 = eT^u + U_0(u, v), \quad v^1 = eT^v + V_0(u, v),$$

where $U_0, V_0$ and their Jacobians are small for all $(u, v)$. In particular let $|V_0| \leq \delta_0$, where $\delta_0 \to 0$ as $\epsilon, \tau \to 0$. In $(u, v)$-coordinates, $M_{+}: v = 0$ is an invariant manifold so that $V_0(u, 0) = 0$. Thus $|V_0(u, v)| \leq \delta_0|v|$ and $|v^1| \geq |eT^v| - \delta_0|v| \geq (b - \delta_0)|v|$, by (2.4). Hence $|v^n| \geq (b - \delta_0)^n|v|$, where $b - \delta_0 > 1$ if $\delta_0$ is sufficiently small. If $(x, y) \in M_{+}$ and $(u, v)$ is given by (3.4), then $v \neq 0$ and $|v^n| \to \infty$ as $n \to \infty$. The $R_0^{-1}$ image of $(u^n, v^n)$ is $(x^n, y^n)$ and $y^n = v^n + \eta_0(u^n)$. The boundedness of $\eta_0(u)$ gives $|y^n| \to \infty$ as $n \to \infty$. For large $n$, (2.6) and (2.3) show that $|x^n| = |eT^x|^{n-1} \leq a|x^{n-1}|$. Hence $|x^n| \to 0$ as $n \to \infty$. This gives the relations (3.2). The relations (3.3) are verified in the same way.

4. Consider the problem of linearizing $T^1$, that is, the problem of the existence of a continuous, one-to-one map of the $(x, y)$-space onto the $(u, v)$-space

$$(4.1)\quad R_1: \ u = U_1(x, y), \quad v = V_1(x, y)$$

such that

$$(4.2)\quad L^1R_1 = R_1T^1.$$
The relation (4.2) is equivalent to a functional equation

\[ e^x V_1(x, y) = V_1(e^x + X(1, x, y), e^y + Y(1, x, y)) \]

for \( V_1 \) and a similar one for \( U_1 \). In order to obtain a solution for \( V_1 \), consider the successive approximations

\[ V(x, y) = y, \]

\[ V_n(x, y) = e^{-A} V_{n-1}(e^x + X(1, x, y), e^y + Y(1, x, y)) \]

for \( n \geq 1 \). If \( r, e \) in (2.6), (2.7) are sufficiently small, a simple induction shows the existence of \( K, \delta, \eta \), with \( 0 < \delta, \eta < 1 \), such that \( |V_n - V_{n-1}| \leq K \eta^n |x| + |y| \) for \( n = 1, 2, \ldots ; [2, p. 613] \). Hence \( V_1(x, y) = \lim V_n(x, y), n \to \infty \), exists uniformly on bounded \((x, y)\)-sets. The existence of a continuous \( U_1(x, y) \) is obtained similarly.

It follows that there is a continuous map (4.1) satisfying (4.2). It therefore only remains to show that \( R_1 \) is one-to-one and onto the \((u, v)\)-space. Note that from (2.4), (2.6), (4.4a) and corresponding formulae for the successive approximations of \( U_1 \), it is seen that

\[ R_1(0, 0) = (0, 0), \]

\[ U_1(x, y) = x \text{ if } |x| \geq r/a \text{ and } V_1(x, y) = y \text{ if } |y| \geq r. \]

Suppose, if possible, that there exists a pair of points \((x_1, y_1), (x_2, y_2)\) satisfying

\[ R_1(x_1, y_1) = R_1(x_2, y_2) \text{ but } (x_1, y_1) \neq (x_2, y_2). \]

It follows from (4.2) and a simple induction that

\[ R_1(x_1^n, y_1^n) = R_1(x_2^n, y_2^n) \quad \text{for } n = 0, \pm 1, \ldots, \]

where \((x_1^n, y_1^n) = T^n(x_1, y_1)\). Since \( T \) is one-to-one by (2.3) and (2.7),

\[ (x_1^n, y_1^n) \neq (x_2^n, y_2^n) \quad \text{for } n = 0, \pm 1, \ldots. \]

Consider first the case (i) that either \((x_1, y_1)\) or \((x_2, y_2)\) is on an unstable or stable manifold \( M_{\pm} \), say, \((x_1, y_1) \in M_+\). Then \((x_2, y_2) \in M_+\). For otherwise, \((x_1^n, y_1^n) \to (0, 0)\) and, by (3.2), \( |y_2^n| \to \infty \), as \( n \to \infty \), so that \( R_1(x_1^n, y_1^n) \to 0 \), by (4.5), and \( |R_1(x_2^n, y_2^n)| \to \infty \), by (4.6), as \( n \to \infty \). This contradicts (4.8). Now \((x_j, y_j) \in M_+\) for \( j = 1, 2 \) implies that \( y_1^j = \eta_0(x_1^j) \) for \( j = 1, 2 \), where \( M_+: y = \eta_0(x) \). It follows from (3.1), (3.3) that \( |x_1^n| \to \infty \) as \( n \to -\infty \). But then (4.6) and (4.8) imply that \( x_1^n = x_2^n \) if \( -n \) is large. Hence \( y_1^n = \eta_0(x_1^n) \) is the same as \( y_2^n = \eta_0(x_2^n) \) for large \(-n\). This contradicts (4.9) and shows that case (i) cannot occur.
Consider the case (ii) that neither \((x_1, y_1)\) nor \((x_2, y_2)\) is on \(M_\pm\). Then \(|x_j^n| \to 0\), \(|y_j^n| \to \infty\) as \(n \to \infty\) and \(|x_j^n| \to \infty\), \(|y_j^n| \to 0\) as \(n \to -\infty\) for \(j = 1, 2\) (cf. (3.2)–(3.3)). The argument in [2, p. 614] shows that case (ii) is impossible if \(a, b, \epsilon\) in (2.4), (2.7) satisfy \(a+2\epsilon < 1 < b-2\epsilon\). Hence \(R_1\) is one-to-one. It is an open map by invariance of domain. It is also easy to see from (4.6) that the range of \(R\) is closed. Hence \(R\) is onto.

5. It was observed by Sternberg [3, p. 817] that (4.2) implies that the map

\[
R = \int_0^1 L^{-t} R_1 T^t dt
\]

satisfies, for all \(t\),

\[
L' R = R T'.
\]

Thus, in order to complete the proof of (*) i.e., of (1.8), it only remains to verify that the map (5.1) is one-to-one and onto.

In view of the normalizations above for \(T^t, 0 \leq t \leq 1\), this verification turns out to be the same as the verification that \(R_1\) is one-to-one. In fact, if (5.1) is written as

\[
R: u = U(x, y), \quad v = V(x, y),
\]

then

\[
R(0, 0) = (0, 0).
\]

Also, by (5.1) and (2.3), (2.6), (4.8), there is a constant \(c > 0\) such that

\[
U(x, y) = x \text{ if } |x| \geq c, \quad V(x, y) = y \text{ if } |y| \geq c.
\]

Thus, analogues of the relations (4.2), (4.5), (4.6) which were used to show that \(R_1\) is one-to-one hold for \(R\), namely, (5.2), (5.4), (5.5). It follows that \(R\) is one-to-one and onto. This completes the proof of (*).

References


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