ON CERTAIN SEQUENCE TO SEQUENCE TRANSFORMATIONS WHICH PRESERVE CONVERGENCE

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1. Introduction. The purpose of this paper is to consider the following question. For what subsets \( U \) of the set \( S \) of all complex number sequences is it true that any triangular matrix \( A \) \((A_{pq} = 0 \text{ if } q > p)\) which is convergence preserving over \( U \) is convergence preserving? Several subsets of \( S \) having this property are given here. The methods of proof generally avoid the three well-known conditions of Silverman and Toeplitz, namely,

\[
\begin{align*}
(1) \quad & \{A_{nk}\}_{n=1}^\infty \text{ converges,} \\
(1.1) \quad & \left\{ \sum_{p=1}^{n} A_{np} \right\}_{n=1}^\infty \text{ converges,} \\
(3) \quad & \text{there exists } M \text{ such that } \sum_{p=1}^{n} |A_{np}| < M, \quad n = 1, 2, 3, \ldots,
\end{align*}
\]

which are necessary and sufficient for \( A \) to be convergence preserving.

The obvious method of attack when a given subset \( U \) of \( S \) is being considered would seem to be to try to prove that if \( A \) is convergence preserving over \( U \), then \( A \) satisfies (1.1). This method has neither been useful in making conjectures nor in proving them, except in one case. The more fruitful approach is to utilize the observations listed in the next section.

2. Preliminaries. The following notation will be used:

\( S_c \): set of all convergent complex sequences.
\( S_{ck} \): set of all real sequences convergent to \( k \).
\( \{k\} \): the sequence \( \{a_p\} \) such that \( a_i = k, \quad i = 1, 2, 3, \ldots \).
\( T \): set of all sequences \( \{a_p\} \) such that either \( \{a_p\} \in S_c \) or \( \{a_p\} \in S_{ck} \).
\( S_{ck+} \): set of all real sequences \( \{a_p\} \) convergent to \( k \) such that \( a_i \leq k, \quad i = 1, 2, 3, \ldots \).
\( S_{ck-} \): set of all real sequences \( \{a_p\} \) convergent to \( k \) such that \( a_i \geq k, \quad i = 1, 2, 3, \ldots \).
\( S_{BV} \): set of all sequences \( \{a_p\} \) such that \( \sum |a_{p+1} - a_p| \) converges.

**Definition.** The triangular matrix \( A \) is convergence preserving (abbreviated: \( A \) is c.p.) means if \( x \in S_c \), then \( Ax \in S_c \).

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DEFINITION. If \( U \) is a subset of \( S \) which contains an element of \( S \), then the statement that the triangular matrix \( A \) is convergence preserving over \( U \) (abbreviated: \( A \) is c.p.o. \( U \)) means if \( x \in U \cdot S \), then \( A x \in S \).

DEFINITION. Suppose \( \{a_p\} \) converges to \( a \) and \( \{b_p\} \) converges to \( b \). The statement that \( \{b_p\} \) converges faster (slower) than \( \{a_p\} \) means \( \lim (b_p - a)/(a_p - a) = 0 \) (\( \lim (a_p - a)/(b_p - b) = 0 \)). (Compare with [2, p. 128].)

As indicated in the introduction, most of the proofs in the next section depend on the following trivially obvious observations.

OBSERVATION 2.1. The set of all finite linear combinations of elements of \( T \) with complex coefficients is \( S_c \).

OBSERVATION 2.2. If \( A \) is c.p.o. \( T \), then \( A \) is c.p.

PROOF. Follows from Observation 2.1 and the fact that \( A \) is a linear transformation defined over \( S \).

EXAMPLE. \( A \), with \( A_{pq} = 0 \) if \( p \neq q \), \( A_{pq} = (-1)^{p+1} \) if \( p = q \), is c.p.o. \( S_c \), but is not c.p.

OBSERVATION 2.3. If \( A \) is c.p.o. the set of all sequences with positive terms, then \( A \) is c.p.

OBSERVATION 2.4. If \( A \) is c.p.o. \( S_{\alpha} \) and \( k \neq 0 \), then \( A \) is c.p.

OBSERVATION 2.5. If \( A \) is c.p.o. \( S_{\alpha + 1} \) and \( A \{1\} \in S_{\alpha} \), then \( A \) is c.p.

OBSERVATION 2.6. If \( A \) is c.p.o. \( S_{\alpha + 1} \) and \( k \neq 0 \), then \( A \) is c.p.

3. THEOREMS. We begin with a theorem which gives a condition that is generally not strong enough to make \( A \) convergence preserving.

THEOREM 3.1. If \( A \) is c.p.o. \( S_{\alpha + 1} \) and \( k \neq 0 \), then \( A \) is c.p.o. \( S_{\beta + 1} \).

PROOF. Suppose the real sequence \( \{a_p\} \subseteq S_{\beta + 1} \). Let \( b_1 = |a_1| \) and \( b_n = |a_1| + \sum_{p=1}^{n-1} |a_{p+1} - a_p| \), \( n = 2, 3, 4, \ldots \), and let \( c_n = b_n - a_n \), \( n = 1, 2, 3, \ldots \). Then \( \{b_p\} \) converges, say to \( b \), and \( \{c_p\} \) converges, say to \( c \). Let \( b'_n = b + k - b_n \) and \( c'_n = c + k - c_n \), \( n = 1, 2, 3, \ldots \). Then \( \{b'_p\} \subseteq S_{\alpha + 1} \), \( \{c'_p\} \subseteq S_{\alpha + 1} \), and \( a_n = c'_n - b'_n + b - c, n = 1, 2, 3, \ldots \). Therefore, \( A \{a_p\} \subseteq S_{\alpha} \), since \( A \{b - c\} = [(b - c)/k]A \{k\} \). The complex case follows from the fact that if \( \{x_p\} \subseteq S_{\beta + 1} \) and \( x_p = x_p + iy_p, p = 1, 2, 3, \ldots \), where each of \( \{x_p\} \) and \( \{y_p\} \) is a real sequence, then \( \{x_p\} \subseteq S_{\beta + 1} \) and \( \{y_p\} \subseteq S_{\beta + 1} \).

COROLLARY 3.1a. If \( A \) is c.p.o. \( S_{\alpha + 1} \) and \( A \{1\} \subseteq S_{\alpha} \), then \( A \) is c.p.o. \( S_{\beta + 1} \).

EXAMPLE. Corollary 3.1a is the best possible in a certain sense as shown by the following example: Let \( A_{pq} = 0 \) if \( q > p \), \( A_{pq} = (-1)^{p+q} \).
Theorem 3.2. Suppose $X$ is a real sequence which converges to 1. If $A$ is convergence preserving over the set of all real sequences $\{a_p\}$ convergent to 1 such that $|X_p-1| \leq |a_p-1|$, $p = 1, 2, 3, \cdots$, then $A$ is c.p.

Proof. Suppose $\{a_p\} \subseteq S_{00}$. If $p$ is a positive integer, let $b_p = X_p$ if $(X_p-1)a_p \geq 0$, $b_p = 2 - X_p$ if $(X_p-1)a_p < 0$, and let $c_p = b_p + a_p$. Clearly $|X_p-1| = |b_p-1|$, $|X_p-1| \leq |c_p-1|$, $p = 1, 2, 3, \cdots$, and $\{b_p\}$ and $\{c_p\}$ converge to 1. Therefore $A\{b_p\} \subseteq S_c$ and $A\{c_p\} \subseteq S_c$.

Thus $A\{a_p\} \subseteq S_c$, and therefore $A$ is c.p.o. $S_{00}$. But $\{X_p-1\} \subseteq S_{00}$. Hence $A\{X_p-1\} \subseteq S_c$. Therefore $A\{1\} \subseteq S_c$, since $A\{X_p\} \subseteq S_c$. Hence by Observation 2.2, $A$ is c.p.

We consider the next theorem mainly for comparison with Theorem 3.2.

Theorem 3.3. Suppose $X$ is a real sequence which converges to 1 such that $X_p = 1$ for no $p$. If $A$ is c.p.o. the set of all real sequences $\{a_p\}$ such that $|a_p-1| \leq |X_p-1|$, $p = 1, 2, 3, \cdots$, then $A$ satisfies (1) and (2) of (1.1) but is not necessarily c.p.

Proof. It follows from the hypothesis that $A\{1\} \subseteq S_c$. Hence (2) of (1.1) holds. Thus it is clear that if $\{a_p\}$ is a real sequence such that $|a_p+1-1| \leq |X_p-1|$, $p = 1, 2, 3, \cdots$, then $A\{a_p\} \subseteq S_c$. Suppose $k$ is a positive integer. Let $a_p = 0$ if $p \neq k$ and $a_k = (1/2)|X_k-1|$. Then $A\{a_p\} \subseteq S_c$. Hence $\{A\{a_p\}\}_{n=1}^\infty$ converges, and so (1) of (1.1) holds. To see that $A$ is not necessarily convergence preserving, consider the matrix $A$ defined in the example following Corollary 3.1a. Let $X_p = 1 + 1/p$, $p = 1, 2, 3, \cdots$. Since $A\{1\} \subseteq S_c$, it is clear that if $\{a_p\}$ is a real sequence such that $|a_p-1| \leq |X_p-1|$, $p = 1, 2, 3, \cdots$, then $A\{a_p\} \subseteq S_c$. But $A$ is not c.p.

Theorem 3.4. Suppose $X$ is a real sequence which converges to 1. If $A$ is c.p.o. the set of all real sequences which converge to 1 slower than $X$, then $A$ is c.p.

Proof. Let $Y_p = |X_p-1|^{1/2}+1$ if $X_p \neq 1$, $Y_p = 1 + 1/p$ if $X_p = 1$, $p = 1, 2, 3, \cdots$. Then $\{Y_p\}$ converges to 1 slower than $X$. Thus $A\{Y_p\} \subseteq S_c$. If $\{a_p\}$ is a real sequence which converges to 1 such that for each positive integer $p$, $|Y_p-1| \leq |a_p-1|$, then $\{a_p\}$ converges slower than $X$, and so $A\{a_p\} \subseteq S_c$. Hence by Theorem 3.2, $A$ is c.p.
THEOREM 3.5. If $A$ is c.p.o. the set of all real sequences $\{a_p\}$ such that $|a_i| \geq |a_{i+1}|$, $i = 1, 2, 3, \cdots$, then $A$ is c.p.

Proof. Let $a_p = 1$ if $p = 1$, $a_p = 0$ if $p > 1$. Then $\{A_{a_1}\} = A\{a_p\} \subseteq S_e$. Let $b_p = 1$ if $p = 1$ or 2, $b_p = 0$ if $p > 2$. Then $A\{b_p\} \subseteq S_e$ and so $\{A_{a_2}\} \subseteq S_e$ since $A\{b_p\} = \{A_{a_1} + A_{a_2}\}$. Hence by an obvious induction, (1) of (1.1) holds. Clearly (2) of (1.1) holds since $A\{1\} \subseteq S_e$. Suppose (3) of (1.1) does not hold. Let $A = B + IC$, where $B$ and $C$ are real triangular matrices ($B_{pq} = C_{pq} = 0$ if $q > p$). Then either $B$ or $C$ fails to satisfy a condition comparable to (3) of (1.1); let us say $B$. Clearly $B$ satisfies conditions comparable to (1) and (2) of (1.1). Using only a trivial modification of a proof of Hardy [1, p. 46], a real sequence $\{y_p\}$ can be found such that $\|y_p\| \in S_{e+1}$ and $B\{y_p\}$ contains a subsequence divergent to $+\infty$. Therefore (3) of (1.1) holds for $A$, and the proof is complete.

The proof of Theorem 3.5 can be modified slightly to prove the following theorem.

THEOREM 3.6. If $A$ is convergence preserving over the set of all sequences $\{a_p\}$ such that $|a_i| > |a_{i+1}|$, $i = 1, 2, 3, \cdots$, then $A$ is c.p.

Corollary 3.6a. If $A$ is c.p.o. the set of all sequences $\{a_p\}$ such that $a_i \neq a_{i+1}$, $i = 1, 2, 3, \cdots$, then $A$ is c.p.

THEOREM 3.7. If $A$ is convergence preserving over the set of all sequences whose terms are elements of the Cantor set, then $A$ is c.p.

Proof. We note that $A\{1\} \subseteq S_e$ since 1 is an element of the Cantor set. Suppose $\{a_p\} \subseteq S_{e+4}$. Let $k$ be a positive number such that $ka_p < 1$, $p = 1, 2, 3, \cdots$, and let $b_p = ka_p$, $p = 1, 2, 3, \cdots$. Suppose $n$ is a positive integer. Let $b_n = .b_{n1}b_{n2}b_{n3} \cdots$ in ternary fractional form. For each positive integer $p$, let $c_{np} = 0$ if $b_{np} = 0$ or 2, $c_{np} = 1$ if $b_{np} = 1$. Let $d_{np} = b_{np} - c_{np}$, $p = 1, 2, 3, \cdots$, and denote $\cdot c_{n1}c_{n2}c_{n3} \cdots$ and $\cdot d_{n1}d_{n2}d_{n3} \cdots$ in ternary fractional form by $c_n$ and $d_n$, respectively. We note that $d_n$ and $2c_n$ are both elements of the Cantor set, and $a_n = k^{-1}(c_n + d_n)$. Since $\{c_p\}$ and $\{d_p\}$ both converge, we see that $A\{a_p\} \subseteq S_e$. Hence by Observation 2.5, $A$ is c.p.

REFERENCES