potents in each semigroup was a finite discrete set. It might be of interest to know if there exists a semigroup $S = ESE$ which is compact connected, has a zero, is not acyclic and such that the set of idempotents is connected.

**Bibliography**


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**CONFORMAL VECTOR FIELDS IN COMPACT RIEMANNIAN MANIFOLDS**

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1. **Introduction.** Let $V^n$ be a compact Riemannian manifold of dimension $n$ and of class $C^3$. Let $g_{ij}(x)$ of class $C^2$ be the coefficients of the fundamental metric which is assumed to be positive definite. Let $\Gamma^{ik}_{kj}$ be the Christoffel symbol, $R_{ijkh}$ the curvature tensor and $R_{ij}$ the Ricci tensor.

Let $\phi$ be an arbitrary scalar invariant, $\xi^i$ an arbitrary vector field and $\xi_{i_1i_2\ldots i_p}$ an arbitrary anti-symmetric tensor field of order $p$, all of class $C^2$ in $V^n$. We shall make use of the following results obtained by S. Bochner and K. Yano [1, pp. 31, 51, 69]:

\[(1.1) \quad (\Delta \phi \geq 0 \text{ everywhere in } V^n) \Rightarrow (\phi = \text{constant everywhere in } V^n).\]

\[(1.2) \quad \int_{V^n} \xi^{i_1}_{,i_1} \, dv = 0.\]

\[(1.3) \quad \int_{V^n} (R_{ij} \xi^i_{,j} + \xi^j_{,i} \xi^{i_1}_{,j} - \xi^j_{,i} \xi^{i_1}_{,i}) \, dv = 0.\]

\[(1.4) \quad \int_{V^n} (F\{\xi_{i_1i_2\ldots i_p}\} + \xi^{i_1i_2\ldots i_p}_{,i} \xi^{j_1j_2\ldots j_p}_{,j} - \xi^{i_1i_2\ldots i_p}_{,i} \xi^{j_1j_2\ldots j_p}_{,j}) \, dv = 0\]

where

\[ F\{\xi_{i_1} \cdots \xi_{i_p}\} = R_{ij} \xi^{j} \cdots \xi^{j} \xi_{i_2} \cdots \xi_{i_p} + \frac{\phi - 1}{2} R_{ijkl} \xi^{j} \cdots \xi^{j} \xi^{k} \xi_{i_2} \cdots \xi_{i_p} \]

and \( V^n \) in equations (1.2) to (1.4) is assumed to be orientable.

If a vector field \( \xi^i \) defines a one-parameter continuous group of infinitesimal conformal transformations in \( V^n \), the intrinsic derivative of \( \xi^i dx^i/\theta s \) along any geodesic \( x^i(s) \) depends only on the point and not on the direction of the geodesic passing through the point; that is

\[
\frac{dx^i}{ds} \frac{dx^j}{ds} = \frac{1}{n} \xi^i \xi^j.
\]  

(1.5)

Let \( \xi^i \) be an arbitrary vector field which does not necessarily define a one-parameter continuous group of infinitesimal conformal transformations. Then, instead of (1.5), we have in general

\[
\frac{dx^i}{ds} \frac{dx^j}{ds} = f(x^i, \xi^i, \frac{dx^i}{ds}).
\]

In the following, such \( \xi^i \) will be associated with an arbitrary but fixed scalar field \( \phi \) in a certain way to give a generalization of some well-known vector fields.

2. **Definition.** Let \( \lambda^i \) be an arbitrary unit vector field. Consider \( M \) defined by

\[
M = \xi_{i,j} \lambda^i \lambda^j - \phi.
\]

(2.1)

When \( \xi^i \) defines a one-parameter continuous group of infinitesimal conformal transformations and \( \phi = \xi^i, i/n \), we have, by (1.5), \( M = 0 \). Hence it seems appropriate to call \( M \) the \( \phi \)-conformal measure of \( \xi^i \) with respect to \( \lambda^i \).

Let \( \lambda^i_\alpha, \alpha = 1, 2, \ldots, n \), be \( n \) mutually orthogonal unit vector fields in \( V^n \) and \( M_\alpha \) the \( \phi \)-conformal measures of \( \xi^i \) with respect to them, that is

\[
M_\alpha = \xi_{i,j} \lambda^i_\alpha \lambda^j_\alpha - \phi.
\]

The mean of \( M_\alpha \) is equal to

\[
\frac{1}{n} \sum_{\alpha = 1}^{n} M_\alpha = \frac{1}{n} g^{ij} \xi_{i,j} - \phi
\]

(2.2)

which is evidently independent of the choice of the orthogonal ennumple \( \lambda^i_\alpha \). A vector field \( \xi^i \) for which the mean (2.2) vanishes is
called a \( \phi \)-conformal vector field in \( V^n \) or simply a conformal vector field in \( V^n \). To each \( \phi \) there corresponds one conformal vector field. The definition leads immediately to the following necessary and sufficient condition

\[
\xi_{i, i} = n\phi. \tag{2.3}
\]

Since a harmonic vector field and a Killing vector field satisfy (2.3) when \( \phi = 0 \), a conformal vector field may be considered as a generalization of them in this sense.

We intend to investigate properties of conformal vector fields in a compact Riemannian manifold \( V^n \), particularly the global non-existence of these fields in \( V^n \). A generalization of our concept to tensor fields is given at the end of the paper.

3. Properties. By definition of the Laplacean (denoted by \( \Delta \)) of a scalar field, we obtain for a conformal vector field \( \xi^i \)

\[
\Delta \xi^i = g^{ik} \xi_{i, k} = n\Delta\phi. \tag{3.1}
\]

The tendency of a vector \( v^i \) in a unit direction \( a^i \) is defined as the projection of the vector \( \phi v^i, k \) in the direction of \( a^i \). It is well known that the divergence of a vector in \( V^n \) is the sum of tendencies of the vector for \( n \) mutually orthogonal directions in \( V^n \) and that of a unit vector in \( V^2 \) is numerically equal to the geodesic curvature of a curve normal to the vector [2, p. 422; 3, p. 201]. Hence from (1.1) and (1.2) we have the following two theorems.

**Theorem 3.1.** If \( \phi \) satisfies \( \Delta\phi \geq 0 \) everywhere in a compact Riemannian manifold \( V^n \), then the sum of tendencies of the \( \phi \)-conformal vector field for \( n \) mutually orthogonal directions is constant throughout the manifold. If \( \phi \) satisfies \( \Delta\phi \geq 0 \) everywhere in a compact Riemannian manifold \( V^2 \), then the geodesic curvature of the orthogonal trajectories of the curves of the \( \phi \)-conformal unit vector field is constant throughout the manifold.

**Theorem 3.2.** In a compact orientable Riemannian manifold \( V^n \), there exists no \( \phi \)-conformal vector field with \( \phi > 0 \) or \( \phi < 0 \) everywhere in the manifold and therefore the divergence of a \( \phi \)-conformal vector field is a constant everywhere in \( V^n \) if and only if the constant is zero. In a compact orientable Riemannian manifold \( V^2 \), the orthogonal trajectories of the curves of a \( \phi \)-conformal unit vector field form a family of geodesics in the manifold if and only if the divergence of the vector field is constant throughout.

By (2.3) we may write (1.3) as
\[ \int_{\mathcal{V}} (\xi,_{ij} \xi,_{ij} + R_{ij} \xi^i \xi^j) dv = \int_{\mathcal{V}} n^2 \phi^2 dv \]

and

\[ \int_{\mathcal{V}} (\xi,_{ij} \xi,_{i} - n^2 \phi^2) dv = - \int_{\mathcal{V}} R_{ij} \xi^i dv \]

which lead to the following two theorems:

**Theorem 3.3.** In a compact orientable Riemannian manifold \( \mathcal{V} \) there exists no \( \phi \)-conformal vector field \( \xi^i \) which satisfies

\[ \int_{\mathcal{V}} (R_{ij} \xi^i \xi^j + \xi,_{ij} \xi,_{i}) dv = 0 \]

unless we have \( \phi = 0 \) and then automatically the equality sign holds.

**Theorem 3.4.** In a compact orientable Riemannian manifold with negative (positive) definite Ricci curvature throughout, there exists no \( \phi \)-conformal vector field other than zero vector field which satisfies

\[ \int_{\mathcal{V}} (\xi,_{ij} \xi,_{i} - n^2 \phi^2) dv \leq 0 \quad (\geq 0). \]

The above theorem includes as special cases some results about global nonexistence of harmonic vector field and Killing vector field obtained by S. Bochner [1, pp. 37, 39].

4. **Generalization.** An anti-symmetric tensor field \( \xi,_{i_1i_2 \ldots i_p} \) of order \( p \) is a conformal Killing tensor field if and only if

\[ \frac{dx^i}{ds} \frac{dx^j}{ds} = \phi_{i_1 \ldots i_p} \]

where

\[ \phi_{i_1 \ldots i_p} = \frac{1}{n} \epsilon^{i_1i_2 \ldots i_p} \xi_{i_1i_2 \ldots i_p} \]

is an anti-symmetric tensor of order \( p-1 \) [1, p. 73].

Denote by \( \lambda_a |^i n \) mutually orthogonal unit vector fields, by \( \phi_{i_1i_2 \ldots i_p} \) an arbitrary but fixed anti-symmetric tensor field of order \( p-1 \), and by \( M_a |_{i_1 \ldots i_p} \) the following tensor fields

\[ M_a |_{i_1 \ldots i_p} = \xi_{i_1i_2 \ldots i_p} \lambda_a |^i \lambda_a |^j - \phi_{i_1 \ldots i_p}. \]

The mean of (4.2) is equal to
\begin{equation}
\frac{1}{n} \sum_{a=1}^{n} M_a |_{i_1 \ldots i_p} = \frac{1}{n} \xi_{i_1 \ldots i_p, j} g^{ij} - \phi_{i_1 \ldots i_p}
\end{equation}

which is independent of the choice of the orthogonal ensemble $\lambda_a$. An anti-symmetric tensor field $\xi_{i_1 \ldots i_p}$ for which (4.3) vanishes is called a $\phi$-conformal tensor field in $V^n$. Thus $\xi_{i_1 \ldots i_p}$ is a $\phi$-conformal tensor field in $V^n$ if and only if
\begin{equation}
g^{ij} \xi_{i_1 i_2 \ldots i_p, j} = n\phi_{i_1 \ldots i_p}.
\end{equation}

Obviously, a $\phi$-conformal tensor field includes a conformal Killing tensor field as a special case.

Substituting (4.4) into (1.4) gives

**Theorem 4.1.** In a compact orientable Riemannian manifold there exists no $\phi$-conformal tensor field of order $p$ which satisfies

$\mathcal{F}\{\xi_{i_1 \ldots i_p}\} + \xi_{i_1 i_2 \ldots i_p, j} \xi_{i_1 i_2 \ldots i_p, j} \leq 0$

unless we have $\phi_{i_1 \ldots i_p} = 0$ and then automatically the equality sign holds.

**References**


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