INVERTIBLE ISOTOPIES

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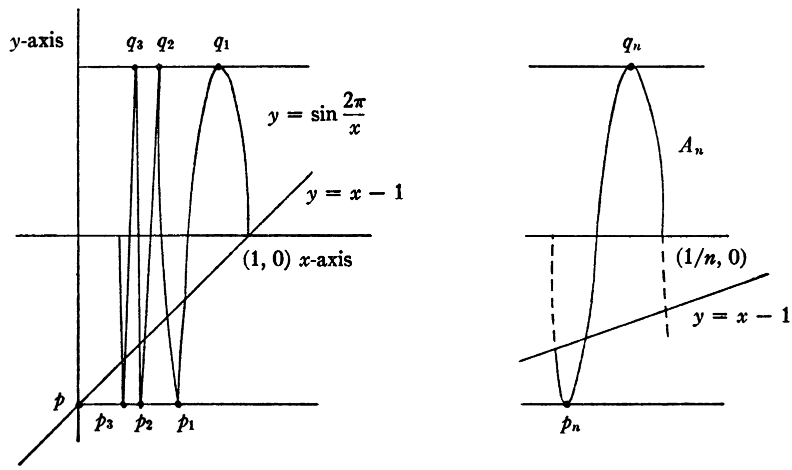
Introduction. The origin of this paper was the remark that "any isotopy of n-dimensional euclidean space obviously possesses a unique extension to an isotopy of the n-sphere." This is true, but not obvious. J. W. Milnor showed me a proof using invariance of domain, and later Herman Gluck, using the results of R. Arens [1], demonstrated that an isotopy of any locally compact, locally connected, Hausdorff space X can be extended to an isotopy of the 1-point compactification $X \cup \{\infty\}$. In Theorem (2.1) we show that local connectedness is not necessary. Local compactness, however, is needed (see Example (1.1) and the first paragraph of §2).

1. Example and theorem. An *isotopy* of a topological space X is a collection $\{h_t\}$, $0 \leqq t \leqq 1$, of autohomeomorphisms of X such that the mapping $h: X \times [0, 1] \to X$ defined by $h(x, t) = h_t(x)$ is continuous. Associated with any isotopy $\{h_t\}$ is also the mapping $H: X \times [0, 1] \to X \times [0, 1]$ defined by $H(x, t) = (h_t(x), t)$. This mapping is one-to-



(a) (b)

FIGURE 1

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one, onto, and continuous. We shall say that an isotopy $\{h_t\}$, $0 \leqq t \leqq 1$, is *invertible* if the collection $\{h_t^{-1}\}$, $0 \leqq t \leqq 1$, of inverse homeomorphisms is an isotopy. Obviously an isotopy $\{h_t\}$ is invertible if and only if the associated mapping H is a homeomorphism.

(1.1) EXAMPLE. Not every isotopy is invertible. Let X be the subspace of the xy-plane consisting of the point $p=(0, -1)$ plus all points (x, y) satisfying $y=\sin(2\pi/x)$, $0<x\leqq 1$. The space X is Hausdorff, but not locally compact. We define two sequences $\{p_n\}$ and $\{q_n\}$ of points in X as shown in Figure 1(a). A collection $\{h_t\}$, $0\leqq t\leqq 1$, of autohomeomorphisms of X is defined as follows:

$$h_0 = h_{1/(2n-1)} = \text{identity}, \qquad n = 1, 2, \cdots.$$

For any t in the closed interval $[1/(2n-1),\ 1/(2n+1)]$, h_t is the identity except on the open arc A_n running from the point $(1/n, 0)$ along X toward the y-axis and ending at the second point of intersection with the line $y=x-1$, see Figure 1(b). This arc is mapped by h_t semi-linearly on itself so that during the interval $[1/(2n-1),$ $1/(2n+1)]$ the point q_n is slid over to p_n and back again without moving any points outside A_n. The timing is chosen so that

$$h_{1/2n}(q_n) = p_n, \qquad n = 1, 2, \cdots.$$

The motion is illustrated in Figure 2 where the graph of $h_t | \overline{A}_n$ is drawn for five different values of t.

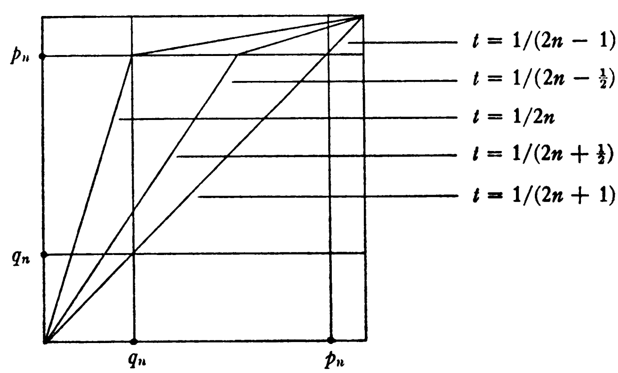

FIGURE 2

The mapping $h: X\times[0, 1]\rightarrow X$ defined by $h(x, t)=h_t(x)$ is obviously continuous except possibly at $(p, 0)$. However, every circular neighborhood of p of radius $\leqq 1$ is mapped into itself by every h_t. Hence h is continuous and $\{h_t\}$ is an isotopy of X. On the other hand, the collection of inverse mappings h_t^{-1} is not an isotopy since $\lim_{n\to\infty} h_{1/2n}^{-1}(p_n) = \lim_{n\to\infty} q_n$ does not exist, but

$$h_0^{-1}\left(\lim_{n\to\infty} p_n\right) = h_0^{-1}(p) = p.$$

(1.2) THEOREM. *Every isotopy $\{h_t\}$ of a locally compact Hausdorff space X is invertible.*

PROOF. We must show that the mapping H defined by $H(x, t) = (h_t(x), t)$ is an autohomeomorphism of $X\times[0, 1]$. The only thing in doubt is the continuity of H^{-1}. We first prove the lemma:

For each $y\in X$, the mapping $t\to h_t^{-1}(y)$ is continuous. Choose $t_0\in[0, 1]$, set $x=h_{t_0}^{-1}(y)$, and select an open neighborhood U of x such that the closure $\overline{U}$ is compact. There then exists a compact neighborhood K of y such that $h_{t_0}^{-1}K\subset U$. We denote the complement of a set in X by means of a prime, and the interior by a small circle. Then,

$$h_{t_0}(U'\cap\overline{U})\subset K', \qquad h_{t_0}(x)\in K^\circ.$$

Since $U'\cap\overline{U}$ and x are compact, K' and K° are open, and H is continuous, there exists a positive number δ such that if $|t-t_0|<\delta$, then

$$h_t(U'\cap\overline{U})\subset K', \qquad h_t(x)\in K^\circ. \tag{1}$$

Assume $|t-t_0|<\delta$. Then $K\subset h_t(U\cup\overline{U}')$ and so

$$h_t^{-1}K\subset U\cup\overline{U}'. \tag{2}$$

Denote the closed interval between t_0 and t by I_t, and the connected set $h(x\times I_t)$ by C. Then $h_t(x)\in C$ and, by (1), $C\subset K^\circ$. Hence, by (2),

$$h_t^{-1}C\subset U\cup\overline{U}'.$$

The sets U and $\overline{U}'$ are open and disjoint. Since $h_t^{-1}C$ is connected and $x\in U\cap h_t^{-1}C$, we conclude that

$$h_t^{-1}C\subset U.$$

But $y=h(x, t_0)\in C$. So we obtain

$$h_t^{-1}(y)\in U, \qquad \text{if } |t-t_0|<\delta,$$

and the proof of the lemma is complete.

We now prove that H^{-1} is continuous. Consider $H(x_0, t_0)=(y_0, t_0)$, and let W be an open neighborhood of (x_0, t_0) in $X\times[0, 1]$. Choose U open in X so that $\overline{U}$ is compact and

$$(x_0, t_0)\in U\times(t_0-\epsilon, t_0+\epsilon)\subset W.$$

Select in X a compact neighborhood K of y_0 such that $h_{t_0}^{-1}K \subset U$. Then, as before,

$$h_{t_0}(U' \cap \overline{U}) \subset K', \qquad h_{t_0}(x_0) \in K^\circ.$$

This time we infer more. There exists a positive δ no bigger than ϵ, and an open subset V of U containing x_0 such that if $|t-t_0| < \delta$, then

$$h_t(U' \cap \overline{U}) \subset K', \qquad h_t V \subset K^\circ.$$

As before, we have

$$h_t^{-1}K \subset U \cup \overline{U}', \qquad \text{if } |t - t_0| < \delta.$$

The set $N = h_{t_0} V$ is an open neighborhood of y_0, and is a subset of K°. We contend that

$$(3) \qquad h_t^{-1}N \subset U, \qquad \text{if } |t - t_0| < \delta.$$

Assume $|t-t_0| < \delta$, and consider an arbitrary element $h_t^{-1}(y)$, $y \in N$. Denote the closed interval between t_0 and t by I_t. By our lemma the set $C = \{h_s^{-1}(y) \mid s \in I_t\}$ is connected. Furthermore for an arbitrary $h_s^{-1}(y) \in C$, we have

$$h_s^{-1}(y) \in h_s^{-1}K^\circ \subset U \cup \overline{U}'.$$

So $C \subset U \cup \overline{U}'$. However, $h_{t_0}^{-1}(y)$ belongs to C and also to $V \subset U$. Since C is connected and U and $\overline{U}'$ are disjoint open sets, we conclude that $C \subset U$. Hence $h_t^{-1}(y) \in U$, and (3) is established. But (3) implies

$$H^{-1}(N \times (t_0 - \delta, t_0 + \delta)) \subset U \times (t_0 - \epsilon, t_0 + \epsilon) \subset W,$$

so the proof of (1.2) is complete.

2. **The 1-point compactification.** It is easy to check that every autohomeomorphism $f: X \to X$ has a unique extension to an autohomeomorphism $\bar{f}$ of the 1-point compactification $X \cup \{\infty\}$ given by $\bar{f} | X = f$ and $\bar{f}(\infty) = \infty$. Can a given isotopy $\{h_t\}$ of X be extended to an isotopy $\{\bar{h}_t\}$ of $X \cup \{\infty\}$? The answer in general is no. For in the example in (1.1), if $\bar{h}(x, t) = \bar{h}_t(x)$, we have

$$\lim_{n\to\infty} \bar{h}(q_n, 1/2n) = \lim_{n\to\infty} p_n = p,$$

$$\bar{h}\left(\lim_{n\to\infty} q_n, \lim_{n\to\infty} 1/2n\right) = \bar{h}_0(\infty) = \infty.$$

However, if X is locally compact and Hausdorff, the answer is yes.

(2.1) Theorem. *Every isotopy $\{h_t\}$ of a locally compact Hausdorff space X has a unique extension to an isotopy of $\overline{X} = X \cup \{\infty\}$.*

PROOF. The question is whether or not the collection $\{\bar{h}_t\}$ of extended homeomorphisms is an isotopy. By Theorem (1.2), the isotopy $\{h_t\}$ is invertible; so the associated mapping H is a homeomorphism. Extend H to $\overline{H}$ by setting

$$\overline{H}(\infty, t) = (\bar{h}_t(\infty), t) = (\infty, t).$$

To prove $\overline{H}$ continuous, consider an open product neighborhood $U\times\Delta$ of (∞, t_0). Then $U'=\overline{X}-U$ is a compact subset of X. Hence $K=H^{-1}(U'\times\bar{\Delta})=\overline{H}^{-1}(U'\times\bar{\Delta})$is compact and therefore closed in the Hausdorff space $\overline{X}\times[0, 1]$. The set $\overline{X}\times\Delta-K$ is open, contains (∞, t_0), and

$$\overline{H}(\overline{X} \times \Delta - K) = \overline{X} \times \Delta - U' \times \bar{\Delta} = U \times \Delta.$$

So $\overline{H}$ is continuous. If π_1 is the projection of $\overline{X}\times[0, 1]$ on $\overline{X}$, the composition $\bar{h}=\pi_1\overline{H}$ is also continuous, and it follows that $\{\bar{h}_t\}$ is an isotopy.

3. **The ideal compactification.** Let X be connected, locally connected, locally compact, Hausdorff, and separable. The *ideal compactification* of Freudenthal [**4**; **5**] is a connected, locally connected, compact, separable, Hausdorff space Y such that X is dense, open, and locally connected in Y (see [**3**, p. 245]) and the set $Y-X$ is 0-dimensional. Furthermore, these properties characterize the compactification (see [**3**, p. 249; **5**, p. 276]).

(3.1) *Every autohomeomorphism f of X has a unique extension to an autohomeomorphism f of the ideal compactification Y.*

PROOF. Let $i: X\rightarrow Y$ be the inclusion mapping. In the terminology of Kelley [**6**] the compactification is the pair (i, Y). Since the pair (if, Y) also satisfied the characteristic properties of the ideal compactification, it is an equivalent compactification. Hence there exists an autohomeomorphism $\bar{f}$ of Y such that $\bar{f}i=if$. The extension $\bar{f}$ is unique because Y is Hausdorff and X is dense in Y.

(3.2) THEOREM. *Every isotopy $\{h_t\}$ of X has a unique extension to an isotopy of the ideal compactification Y.*

PROOF. We must prove that the collection $\{\bar{h}_t\}$ of extended homeomorphisms is an isotopy of Y. By Theorem (1.2) the mapping H associated with $\{h_t\}$ is a homeomorphism, and we define the extension $\overline{H}$ by

$$\overline{H}(x, t) = (\bar{h}_t(x), t).$$

As in the proof of (2.1), it suffices to prove $\overline{H}$ continuous. Let

$p \in Y-X$ and $\bar{H}(p, t_0)=(q, t_0)$, and consider an arbitrary neighborhood W of (q, t_0). Since $q \in Y-X$ which is 0-dimensional, there is an open neighborhood U of q in Y and an open interval Γ about t_0 such that $U \times \Gamma \subset W$ and $\dot{U}=\bar{U}-U \subset X$. The boundary $\dot{U}$ is closed and therefore compact. Hence $K=H^{-1}(\dot{U}\times\bar{\Gamma})$ is compact and therefore closed in $Y\times[0, 1]$. Choose an open product neighborhood $V\times\Delta$ about (p, t_0) such that

$$V \times \Delta \subset Y \times [0, 1] - K,$$

Δ is an open subinterval of Γ,

$$\bar{h}_{t_0} V \subset U,$$

$V \cap X$ is connected (X is locally connected in Y).

Since $V\cap X\times\Delta\subset X\times\Gamma-H^{-1}(\dot{U}\times\bar{\Gamma})$, we have

$$H(V \cap X \times \Delta) \subset X \times \Gamma - \dot{U} \times \bar{\Gamma} \subset U \cup \bar{U}' \times \Gamma.$$

The image $H(V\cap X\times\Delta)$ is connected, the sets U, $\bar{U}'$ are open and disjoint, and h_{t_0} maps points of $V\cap X$ into U. We conclude that

$$H(V \cap X \times \Delta) \subset U \times \Gamma.$$

Consider finally any point $(r, t)\in(V\cap(Y-X))\times\Delta$. Suppose $\bar{h}_t(r) \in\bar{U}'$ (it belongs either to U to $\bar{U}'$). Since $\bar{h}_t$ is continuous and X is dense in Y, there then exists a point of $V\cap X$ which is also mapped by $\bar{h}_t$, and so by h_t, into $\bar{U}'$. This is impossible. We conclude that

$$\bar{H}(V \times \Delta) \subset U \times \Gamma \subset W,$$

and the proof is complete.

4. **Conclusion.** In this section X is always a locally compact Hausdorff space. The compact-open topology on the set $A(X)$ of all autohomeomorphisms of X defines a topological space which we denote by $A(X)_{co}$. If $\{h_t\}$, $0\leqq t\leqq 1$, is an isotopy of X, then the mapping h^*: $[0, 1]\to A(X)_{co}$, defined by $h^*(t)=h_t$, is continuous. Conversely, any path in $A(X)_{co}$ defines an isotopy. (See R. H. Fox [**2**].) In [**1**] Arens considers what he calls the g-topology on $A(X)$, which is larger (more open sets) than the compact-open topology. For any two subsets F, $W\subset X$, let (F, W) denote the set of all $f\in A(X)$ such that $fF\subset W$. The g-topology is defined by taking for a sub-basis all sets (F, W) where F is closed, W is open, and either F or $X-W$ is compact. In the group $A(X)_{co}$ multiplication (composition) is continuous, but the operation $f\to f^{-1}$ generally is not. Arens proves that the g-topology is the smallest admissible topology making $A(X)$ a topological group. He also proves that if X is locally connected, then

$A(X)_{co} = A(X)_g$. It is easy to verify that an invertible isotopy is equivalent to a path $[0, 1] \to A(X)_g$. Combining the last two facts, we obtain Gluck's proof of Theorem (1.2) for a locally connected X. Arens also gives an example of a locally compact Hausdorff space for which the co and g topologies are *not* equal, or equivalently, the identity mapping $A(X)_{co} \to A(X)_g$ is not continuous. In view of Aren's example and theorem (1.2), we have the interesting result that for a locally compact Hausdorff space X and path $[0, 1] \to^{h^*} A(X)_{co}$, the composition $[0, 1] \to^{h^*} A(X)_{co} \to^{\mathrm{id}} A(X)_g$ is necessarily continuous even though $A(X)_{co} \to^{\mathrm{id}} A(X)_g$ is not. Actually, as Gluck observed, this result remains true if the path h^* is replaced by any continuous mapping $h^*: Y \to A(X)_{co}$ of an arbitrary locally connected topological space Y. The reason is that Theorem (1.2) remains true and the same proof works if in the definition of isotopy the interval $[0, 1]$ is replaced by Y. Thus the possible discontinuities of $\mathrm{id}: A(X)_{co} \to A(X)_g$ cannot be detected by any continuous mapping of a locally connected space into $A(X)_{co}$.

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