BOUNDS FOR THE SOLUTIONS OF A CLASS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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1. Let $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ denote the $n$-dimensional Laplace operator and let the symbols $D_r$ and $S_r$ stand for the open sphere $x_1 + \cdots + x_n < r^2$ ($r > 0$) and its boundary $x_1 + \cdots + x_n^2 = r^2$, respectively. We are concerned here with functions $u = u(P)$ ($P \in D_r$) which are of class $C^2$ in $D_r$ and satisfy there the differential equation

$$\Delta u = f(u),$$

or, more generally, the differential inequality

$$\Delta u \geq f(u).$$

In the literature on the subject [1; 2; 3; 5; 7; 8], two closely related problems are investigated:

(a) What are the conditions to be imposed on the function $f(u)$ in order to guarantee the existence of a bound $\phi(r) = \phi(r, R; f)$ such that

$$u(P) \leq \phi(r, R; f)$$

for $P \in S_r$ if $u$ satisfies (2) in a region $D_R$ with $R > r$?

(b) Under what conditions on $f(u)$ will (2) have no solutions which are of class $C^2$ in the entire $n$-space?

Clearly, the nonexistence of such solutions is assured whenever it can be shown that $\phi(0, R; f) \to -\infty$ for $R \to \infty$.

The most general conditions on $f(u)$ for which the existence of such bounds for the solutions of (2) have been established are [3; 5]: $f(u) > 0$, $f'(u) \geq 0$ for $-\infty < u < \infty$,

$$\int_0^\infty \left[ \int_0^u f(t) dt \right]^{1/2} du < \infty.$$  

In fact, if $f(u) > 0$ and $f'(u) \geq 0$, condition (4) is both necessary and sufficient. It was also shown in [3] and [5] that the problem $u(P) = \max$ is solved by a spherically symmetric solution $\phi(r)$ of (1) for which $\phi(r) \to \infty$ for $r \to R$, i.e., by a solution of the ordinary differential equation
(5) \[ \phi''(r) + \frac{n-1}{r} \phi'(r) = f(\phi) \]

for which \( \phi'(0) = 0 \) and \( \phi(r) \to \infty \) for \( r \to R \). In those cases in which this solution can be found explicitly it is thus possible to determine the exact upper bound (3). An example is the two-dimensional equation \( \Delta u = e^u \), which has the well-known solution

\[ u = 2 \log \frac{\sqrt{8R}}{R^2 - r^2} . \]

It was pointed out by Osserman [5] that an upper bound for \( u(P) \) is given by any spherically symmetric function \( v \) of class \( C^2 \) which satisfies the differential inequality

(6) \[ \Delta v \leq f(v) \]

and tends to \( \infty \) as \( r \to R \). We shall here use this remark to find explicit upper bounds for certain classes of functions \( f \). The following statement also gives a lower bound for \( \max u(P) \), which is obtained with the help of a suitable function satisfying the inequality (2).

**Theorem I.** Let \( f(u) \) be a positive, nondecreasing, differentiable function in \( (-\infty, \infty) \), for which

\[ \int_u^\infty \frac{dt}{f(t)} \] \hspace{1cm} (\( u > -\infty \))

exists, and for which

(7) \[ [f(u)]^{1+\lambda} \int_u^\infty \frac{dt}{f(t)} \]

is a nondecreasing function of \( u \) for some non-negative \( \lambda \). If

(8) \[ \phi(r) = \sup_{P \in D_r} u(P) , \]

where \( u(P) \) ranges over all functions of class \( C^2 \) in \( D \), which satisfy (2), then

(9) \[ \frac{c(\lambda)(R^2 - r^2)^2}{R^2} \leq \int_{\phi(r)}^\infty \frac{dt}{f(t)} \leq \frac{R^2 - r^2}{2n} , \]

where

(10) \[ c(\lambda) = \frac{1}{4n} \] \hspace{1cm} (\( 4\lambda \leq n - 2 \))
and

\[ c(\lambda) = \frac{1}{8(2\lambda + 1)} \quad (4\lambda > n - 2). \]

The left-hand inequality (9) (which yields the upper bound for \( \phi(r) \)) is sharp in the sense that for each number of dimensions \( n (n \geq 2) \), there exists an equation (1) with a spherically symmetric solution \( \phi(r) \) for which the sign of equality holds.

The condition that (7) be a nondecreasing function of \( u \) is equivalent to the inequality

\[ f'(u) \int_{u}^{\infty} \frac{dt}{f(t)} \leq 1 + \lambda. \]

It is worth noting that this inequality is always satisfied, for \( \lambda = 0 \), if \( \log f(u) \) is a convex function of \( u \). Indeed, since \( f'/f \) is in this case a nondecreasing function of \( u \), we have

\[ \frac{1}{f(u)} = \int_{u}^{\infty} \frac{f'(t)}{f^2(t)} dt \geq \frac{f'(u)}{f(u)} \int_{u}^{\infty} \frac{dt}{f(t)}, \]

and the assertion follows. This implies the following special result.

If \( \log f(u) \) is a convex nondecreasing function in \((-\infty, \infty)\), and \( \phi(r) \) is defined as before, then

\[ \frac{(R^2 - r^2)^2}{4nR^2} \leq \int_{\phi(r)}^{\infty} \frac{dt}{f(t)} \leq \frac{R^2 - r^2}{2n}. \]

In particular,

\[ \frac{R^2}{4n} \leq \int_{\phi(0)}^{\infty} \frac{dt}{f(t)} \leq \frac{R^2}{2n}. \]

In the case of a solution \( u \) of

\[ \Delta u = e^u \]

which is regular in \( D_R \), (13) shows that

\[ \log \frac{2n}{R^2 - r^2} \leq \phi(r) \leq 2 \log \frac{2\sqrt{nR}}{R^2 - r^2} \]

and, for \( r = 0 \),

\[ 2 \log \frac{\sqrt{(2n)}}{R} \leq \phi(0) \leq 2 \log \frac{2\sqrt{n}}{R}. \]
As already mentioned, the right-hand inequality (15) becomes an equality in the two-dimensional case. For \( n \geq 3 \), no explicit solutions of (14) are known. However, it follows from the fact that the substitution of \( \rho r \) for \( r \) and \( u - 2 \log \rho \) for \( u \) \((\rho > 0)\) transforms the equation into itself, that

\[
\phi(0) = 2 \log \frac{K_n}{R},
\]

where \( K_n \) is a constant. (16) shows that \( \sqrt{(2n)} \leq K_n \leq 2 \sqrt{n} \). For \( n = 3 \), an improved lower bound for \( \phi(0) \) can be obtained from the observation that the 3-sphere of radius \( R \) is contained in the right circular cylinder of the same radius. Hence, \( K_3 < K_2 \), and thus \( 2\sqrt{2} < K_3 < 2\sqrt{3} \).

2. Turning now to the proof of the left-hand inequality (9), we consider the function \( v = v(r) \) defined by

\[
\frac{c}{R^2} (R^2 - r^2)^{2} = \int_{\nu}^{\infty} \frac{dt}{f(t)}.
\]

We evidently have \( v'(0) = 0 \), and \( v(r) \) increases to \( \infty \) as \( r \to R \). If we can show that \( v \) satisfies the differential inequality (6), it will therefore follow that \( \phi(r) \leq v(r) \), and this will establish the left-hand inequality (9). To verify (6), we write \( x \) for any of the variables \( x_1, \ldots, x_n \), and we differentiate (17) twice with respect to \( x \). This yields

\[
-\frac{4c}{R^2} (R^2 - r^2) = -\frac{v_x}{f(v)},
\]

\[
-\frac{4c}{R^2} (R^2 - r^2) + \frac{8c x^2}{R^2} = -\frac{v_{xx}}{f(v)} + \frac{v_x f'(v)}{f^2(v)}
\]

\[
= -\frac{v_{xx}}{f(v)} + \frac{16c^2 x^2}{R^4} (R^2 - r^2)^2 f'(v).
\]

Summing over all the \( x \), we obtain

\[
-\frac{4cn}{R^2} (R^2 - r^2) + \frac{8cr^2}{R^2} = -\frac{\Delta v}{f(v)} + \frac{16c^2 r^2}{R^4} (R^2 - r^2)^2 f'(v),
\]

or, in view of (17),

\[
\frac{\Delta v}{f(v)} = \frac{16c^2 r^2}{R^2} f'(v) \int_{\nu}^{\infty} \frac{dt}{f(t)} + \frac{4nc}{R^2} (R^2 - r^2) - \frac{8cr^2}{R^2}.
\]
Condition (12) therefore leads to the inequality

\[
\frac{\Delta v}{f(v)} \leq 4c \left[ n - \frac{r^2}{R^2} (n - 2 - 4\lambda) \right].
\]

If \(4\lambda \leq n - 2\), it follows that \(\Delta v \leq 4ncf(v)\), and \(v\) will satisfy (6) if \(c\) is given the value (10). If \(4\lambda > n - 2\), the maximum of the right-hand side of (19) (for \(0 \leq r \leq R\)) is attained for \(r = R\), and the value (11) for \(c\) again leads to a function for which (6) holds.

The sign of equality in (9) will hold if \(v\) is a solution of \(\Delta v = f(v)\). Since (19) was obtained from (18) by the use of the inequality (12), this is possible only if (12) becomes an equality. This will occur if

\[
f(u) = u^{1+1/\lambda}, \quad \lambda > 0,
\]

and, for \(\lambda = 0\), if

\[
f(u) = e^u.
\]

Furthermore, the right-hand side of (19) will be equal to the constant 1 only if the coefficient of \(r^2\) vanishes (and, of course, if \(c\) is chosen in accordance with (10)). We thus must have \(4\lambda = n - 2\). Hence, the left-hand inequality (9) will become an equality in the case of the equation

\[
\Delta u = u^{(n+2)/(n-2)}, \quad n \geq 3,
\]

and, if \(n = 2\), the equation (14). The solution of (20) obtained in this way is easily confirmed to be of the form

\[
\begin{align*}
u &= \left[ \frac{R \sqrt{(n(n - 2))}}{R^2 - r^2} \right]^{8/(n-2)}.
\end{align*}
\]

This, incidentally, seems to be the only \(n\)-dimensional equation of the form \(\Delta u = u^k\), \(k > 1\), for which a solution can be obtained in terms of elementary functions.

It should be remarked here that, strictly speaking, the equation \(\Delta u = u^k\) is not covered by Theorem I, since the conditions on \(f(u)\) are satisfied only for \(u > 0\). It is, however, clear that the left-hand inequality (9) will remain valid for solutions of this equation which are positive in \(D_R\). It is also possible to give a more general version of Theorem I which applies to cases in which the hypotheses on \(f(u)\) are satisfied only for \(u > \alpha\), where \(\alpha\) is a given number. Before we formulate this generalization, we prove the right-hand inequality (9).
The function $w = w_\rho(r)$ defined by
\[
\frac{\rho^2 - r^2}{2n} = \int_{\infty}^{\infty} \frac{dt}{f(t)}, \quad \rho > R,
\]
is of class $C^2$ in $D_R$, and it satisfies the differential inequality (2). Indeed, differentiating with respect to $x = x_k$, we obtain
\[
- \frac{x}{n} = - \frac{w_x}{f(w)},
\]
\[
- \frac{1}{n} = - \frac{w_{xx}}{f(w)} + \frac{w^2 f'(w)}{f^2(w)}
\]
\[
= - \frac{w_{xx}}{f(w)} + \frac{x^2}{n^2} f'(w),
\]
and, summing over all the $x_k$, we obtain
\[
\frac{\Delta w}{f(w)} = 1 + \frac{r^2}{n^2} f'(w).
\]
Since $f'(w) \geq 0$, it follows that $\Delta w \geq f(w)$, and thus, in view of the results quoted above, $w(r) \leq \phi(r)$. Since $\rho$ may be taken arbitrarily close to $R$, this establishes the right-hand inequality (9). It may also be noted that the only assumption used was $f'(w) \geq 0$; this estimate is therefore valid in the most general case in which the existence of $\phi(r)$ was proved in [3] and [5].

3. We now state the more general version of Theorem I.

**Theorem II.** Let $f(u)$ be continuous in $(-\infty, \infty)$, but satisfy the other hypotheses of Theorem I only for $u > a$, where $a$ is a given number; furthermore, let
\[
\int_{-\infty}^{\infty} \frac{dt}{f(t)} = \infty.
\]
If $u$ is a function of class $C^2$ in $D_r$, satisfying the inequality (2), and if $P \in S_r (r < R)$, then either $u(P) \leq \alpha$ or, if $u = u(P) > \alpha$,
\[
\frac{c(\lambda)}{R^2} (R^2 - r^2)^2 \leq \int_{-\infty}^{\infty} \frac{dt}{f(t)}.
\]
The proof is again based on the fact that the function $v$ defined in (17) tends to $\infty$ for $r \to R$ and satisfies the inequality (6). Condition
(21) guarantees that this definition is meaningful for all given values of $R$ and all $r$ in $[0, R)$. The fact that $u \leq v$, if $u$ satisfies (2), now follows by a slight modification of the argument used in [5]. Since $\Delta v \leq f(v)$, we have

$$\Delta(u - v) \geq f(u) - f(v).$$

Suppose there exists a nonempty set $T$ in $D_R$ on which $u > v$. Since $v > \alpha$, we necessarily have $u > \alpha$ on $T$, and it follows from our assumptions that $f(u) \geq f(v)$ on this set. By (23), $u - v$ is therefore subharmonic on $T$. We may assume that $u$ is of class $C^2$ on $D_R + S_R$; this assumption can then be removed by a standard argument. Since $v \to \infty$ for $r \to R$, we have $u - v \to -\infty$ for $r \to R$, and it follows that the boundary $B$ of $T$ is in $D_R$. Hence, $u - v = 0$ on $B$, which is absurd, since $u - v$ is positive and subharmonic on $T$. The set $T$ is thus empty, and we must have $u \leq v$ throughout $D_R$. This proves (22).

An immediate consequence of Theorem II is the following result concerning the nonexistence of certain types of entire solutions.

**Theorem III.** If $f(u)$ is subject to the hypotheses of Theorem II, and if there exists a function $u$ satisfying (2) which is of class $C^2$ in the entire space, then

$$u(P) \leq \alpha$$

for all $P$.

Indeed, suppose that $u(P) > \alpha$ for some point $P$. Since (2) remains unchanged under a translation of the coordinate system, we may take $P$ to be the origin. It follows therefore from (22) that

$$c(\lambda) R^2 \leq \int_{u(P)}^\infty \frac{dt}{f(t)},$$

and this produces a contradiction if $R$ is taken large enough.

That entire solutions satisfying (24) may indeed exist is shown by the equation $\Delta u = u^{k+1}$, where $k$ is a positive integer, which satisfies all the assumptions for $\alpha = 0$, and which has the trivial entire solution $u \equiv 0$. A nontrivial example is given by the equation

$$\Delta u = u^2 + u$$

in three dimensions, for which $\alpha = 0$, and which is known to possess a negative entire solution [4; 6].

As a final example, we mention the equation $\Delta u = u^{2k+1}$, where $k$ is
a positive integer. Since the equation remains unchanged if \( u \) is replaced by \( -u \), it follows from Theorem III that, except for the trivial solution \( u = 0 \), the equation has no entire solutions.

References

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8. H. Wittich, Ganze Lösungen der Differentialgleichung \( \Delta u = e^u \), Math. Z. 49 (1943/44), 579–582.

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