COLUMN SEQUENCES IN HAUSDORFF MATRICES

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Corresponding to each sequence \( d \) of complex numbers, the Hausdorff matrix \( H = H(d) \) is given by

\[
H_{n,k} = \begin{cases} 
0 & \text{if } n < k, \\
\binom{n}{k} \Delta^{n-k} d_k & \text{if } n \geq k,
\end{cases}
\]

For convenience, we shall denote the \( k \)th column sequence by \( h^{(k)} \), i.e., \( h^{(k)}_n = H_{n,k}, \ n = 0, 1, \ldots \). For each \( k \geq 1 \), \( C^k \) will denote the \( k \)th power of the Cesàro matrix \((C, 1)\). We shall make use of the fact that, regarded as summability methods, \( C^k \) and \((C, k)\) are equivalent \([1, \text{p. 103}]\). If the \( C^k \) transform (and consequently the \((C, k)\) transform) of a sequence \( s \) has limit \( x \), we shall abbreviate this by \( s_n \rightarrow x (C, k) \).

For Hausdorff matrices which satisfy the condition

\[
\sum_{k=0}^{n} |H_{n,k}| \leq M \quad (M \text{ independent of } n),
\]

it is well known \([1, \text{p. 255}]\) that \( h^{(0)} \) converges and that every other column sequence converges to zero. The purpose of this note is to obtain a weaker form of this result for all Hausdorff matrices for which \( h^{(0)} \) converges.

**Theorem.** If \( H \) is a Hausdorff matrix and \( h^{(0)} \) converges, then \( h^{(k)} \rightarrow 0 \ (C, k) \) for every positive integer \( k \).

**Proof.** The proof depends mainly on the sequence identity

\[
C h^{(k)} = C h^{(k-1)} - \frac{1}{k} h^{(k-1)},
\]

where \( C h^{(k)} \) denotes the \( C \) transform of the sequence \( h^{(k)} \). Noting that, for \( n \geq k \), the \( n \)th term of \( C h^{(k)} \) is

\[
\frac{1}{n+1} \sum_{p=k}^{n} \binom{p}{k} \Delta^{p-k} d_k,
\]

a verification of (1) follows from the identities

\[
\Delta^{p-k} d_k = \Delta^{p-k} d_{k-1} - \Delta^{p-k+1} d_{k-1} \quad \text{and} \quad \binom{p}{k} - \binom{p-1}{k-1} = \binom{p-1}{k-1}.
\]

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If in (1), $k = 1$, convergence of $h^{(0)}$ immediately implies $h^{(1)}_n \to 0$ ($C, 1$). Suppose now that $k - 1$ is a positive integer for which $h^{(k-1)}_n \to 0$ ($C, k - 1$). Applying the $C^{k-1}$ matrix to both sides of (1), we have

$$C^{k}h^{(k)} = C^{k}h^{(k-1)} - \frac{1}{k} C^{k-1}h^{(k-1)}.$$ 

Since $C^{k-1}h^{(k-1)}$ has limit zero, so does $C^{k}h^{(k-1)}$, and therefore $C^{k}h^{(k)}$ has limit zero. This completes the proof.

**Corollary.** If $H$ is a Hausdorff matrix and $h^{(0)}$ converges, then for every positive integer $k$ for which $h^{(k)}$ converges, $h^{(k)}$ has limit zero.

**Proof.** ($C, k$) summability is regular.

**Reference**


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